

Infinite Series

Defn Given a sequence $\{a_k\}$ of real numbers, the symbol $\sum_{k=0}^{\infty} a_k$ is called an infinite series or simply a series. The symbol is meant to suggest an infinite summation $\sum_{k=0}^{\infty} a_k := a_0 + a_1 + a_2 + a_3 + \dots$, and each a_k is called a term or element. For all $n \in \mathbb{N}$, let $s_n = \sum_{k=0}^n a_k = a_0 + a_1 + \dots + a_n$, which is called the n th partial sum. The resulting sequence $\{s_n\}$ is called the sequence of partial sums.

Notes: 1) We may write $\sum a_k$ instead of $\sum_{k=0}^{\infty} a_k$ when the indexing is understood.

2) While we will usually start a series at $k=0$, we could easily start at any $m \in \mathbb{N}$, in which case the series is $\sum_{k=m}^{\infty} a_k = a_m + a_{m+1} + \dots$

3) Notice every series has two sequences associated with it, the sequence of terms $\{a_k\}$ and the sequence of partial sums $\{s_n\}$. We will usually focus our attention on the former (i.e. $\{a_k\}$).

Defn The series $\sum a_k$ is said to converge to S , written $\sum a_k = S$, if the sequence of partial sums $\{s_n\}$ converges to S (i.e. $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k = S$). If $\{s_n\}$ does not converge, we say the series diverges.

 N th Term Test (Cor. 14.5)

If series $\sum a_k$ converges, then $\lim_{k \rightarrow \infty} a_k = 0$.

Note: The contrapositive of this is extremely useful for testing divergence: If $\lim_{k \rightarrow \infty} a_k \neq 0$, then the series diverges.

Defn For a non zero constant $a \in \mathbb{R}$, the series $\sum_{k=0}^{\infty} ar^k$ is called a geometric series, and the number $r \in \mathbb{R}$ is the common ratio or ratio.

Geometric Series Test

For $a \neq 0$, the geometric series $\sum_{k=0}^{\infty} ar^k$ converges to the sum $\frac{a}{1-r}$ when $0 < |r| < 1$, and diverges when $|r| \geq 1$.

Note: The geometric series is one of the few series we can easily compute the sum.

Defn For $p \in \mathbb{R}$, the series $\sum_{k=1}^{\infty} \frac{1}{n^p}$ is called a p-series

p-Series Test

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Defn The series $\sum a_k$ is said to be absolutely convergent or converge absolutely if $\sum |a_k|$ converges.

If $\sum a_k$ converges and $\sum |a_k|$ diverges, then the series $\sum a_k$ is said to be conditionally convergent.

Comparison Test (Thm 14.6)

Let $\sum a_k$ be a series.

- 1) If $\sum a_k$ converges and there exists $M \in \mathbb{R}$ with $|b_k| \leq a_k$ $\forall k > M$, then $\sum b_n$ converges absolutely.
- 2) If $\sum a_k = \infty$ and there exists $M \in \mathbb{R}$ with $a_k \leq |c_k| \forall k > M$, then $\sum c_k$ diverges.

Note: This test is tricky to use because it not only forces you to 'guess' whether the sequence under question converges or diverges, but also relies on your knowledge of convergent and divergent series to compare against.