Math 25
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Sets on Metric Spaces

__Defn__ Let $x_0 \in S$ and $r > 0$. The (open) __neighborhood around x_0 with radius r__, denoted $B_r(x_0)$, is the set

$$B_r(x_0) := \{x \in S : d(x, x_0) < r\}$$

Notes: 1) Some people refer to this set as an __open ball__.
2) These neighborhoods, for any $r > 0$, can be viewed as a basis for all open sets (see below) in S and the fundamental open sets in the metric space (S, d).

__Defn__
Let $E \subseteq S$. An element $x_0 \in E$ is an __interior point__ to E if there exists $r > 0$ such that $B_r(x_0) \subseteq E$. We denote the set of all interior points to E as E^o. The set E is __open__ if every point in E is an interior point (i.e. $E = E^o$).

__Proposition (13.7)__ For any metric space (S, d), we have
1) S is open in S.
2) The empty set $\emptyset$ is open in S.
3) The union of __any__ collection of open sets is open.
4) The intersection of __finite many__ open sets is again an open set.

Note: Any set of points S that satisfies properties (1)-(4) is called a __topology__, regardless on whether there is a metric on S. Topologies are studied in more detail in Math 147.

__Defn__ A subset $E \subseteq S$ is __closed__ if its compliment $S - E$ is an open set. The __closure of $E \subseteq S$__, denoted E^-, is the intersection of all closed sets containing E. The __boundary of $E \subseteq S$__, denoted ∂E, is the set $E^- - E^o$ and the points in ∂E are called __boundary points__.

Proposition 13.9

Let $E \subseteq S$. Then, we have

a) E is closed if and only if $E = E^-$.

b) E is closed if and only if it contains the limit of every convergent sequence $\{s_n\}$ with $s_n \in E$ $\forall n \in \mathbb{N}$.

c) $x \in E^-$ if and only if $\exists \{s_n\}$ with $s_n \in E$ $\forall n \in \mathbb{N}$ such that $\lim_{n \to \infty} s_n = x$.

d) $x \in \partial E$ if and only if $x \in E^-$ and $x \in S - E$.

Note: Some people denote the compliment of E as $E^c := S - E$.

Thm 13.10

Let $\{F_k\}$ be a nested sequence of sets (i.e. $F_{k+1} \subseteq F_k$ $\forall k \in \mathbb{N}$) which are all closed bounded nonempty sets of $\mathbb{R}^n$. Then, $F = \bigcap_{k=1}^{\infty} F_k$ is also closed bounded and nonempty.

Defn

A family $\mathcal{U}$ of open sets is an open cover for a set $E \subseteq S$ if each point in E belongs to at least one set in $\mathcal{U}$ (i.e. $E \subseteq \bigcup \{u : u \in \mathcal{U}\}$)

A subcover of $\mathcal{U}$ is a subfamily of $\mathcal{U}$ which also covers E.

A cover or subcover is finite if only contains a finite number of sets, each of which may contain an infinite number of points.

A set E is compact if every open cover of E has a finite subcover of E.

Proposition 13.13

Every n-cell in $\mathbb{R}^n$ is compact.

Note: An n-cell F is an n-dimensional box $F = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$

In $\mathbb{R}^2$ this is just a rectangle.

Heine-Boral Theorem (13.12)

A set $E \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

Defn A set E in $\mathbb{R}^n$ is bounded if there exists an $M > 0$ where $\max\{|x_j| : j = 1, \ldots, n\} \leq M$ $\forall \vec{x} = (x_1, x_2, \ldots, x_n) \in E$.