Sets on Metric Spaces

Defn. Let $x_0 \in S$ and $r > 0$. The (open) neighborhood around $x_0$ with radius $r$, denoted $B_r(x_0)$, is the set

$$B_r(x_0) = \{ x \in S : d(x, x_0) < r \}$$

Notes: 1) Some people refer to this set as an open ball.
2) These neighborhoods, for any $r > 0$, can be viewed as a basis for all open sets (see below) in $S$ and the fundamental open sets in the metric space $(S, d)$.

Defn. Let $E \subseteq S$. An element $x_0 \in E$ is an interior point to $E$ if there exists $r > 0$ such that $B_r(x_0) \subseteq E$. We denote the set of all interior points to $E$ as $E^o$. The set $E$ is open if every point in $E$ is an interior point (i.e. $E = E^o$).

Proposition (13.7) For any metric space $(S, d)$, we have

1) $S$ is open in $S$.
2) The empty set $\emptyset$ is open in $S$.
3) The union of any collection of open sets is open.
4) The intersection of finite many open sets is again an open set.

Note: Any set of points $S$ that satisfies properties (1)-(4) is called a topology, regardless on whether there is a metric on $S$. Topologies are studied in more detail in Math 197.

Defn. A subset $E \subseteq S$ is closed if its complement $S - E$ is an open set. The closure of $E \subseteq S$, denoted $E^\prime$, is the intersection of all closed sets containing $E$. The boundary of $E \subseteq S$, denoted $\delta E$, is the set $E^\prime - E^o$ and the points in $\delta E$ are called boundary points.
Proposition 13.9
Let \( E \subseteq S \). Then, we have

a) \( E \) is closed if and only if \( E = E^- \).

b) \( E \) is closed if and only if it contains the limit of every convergent sequence \( \{s_n\} \) with \( s_n \in E \ \forall n \in \mathbb{N} \).

c) \( x \in E^- \) if and only if \( \exists \{s_n\} \) with \( s_n \in E \ \forall n \in \mathbb{N} \) such that \( \lim_{n \to \infty} s_n = x \).

d) \( x \in \partial E \) if and only if \( x \in E^- \) and \( x \in S - E \).

Note: Some people denote the complement of \( E \) as \( E^c := S - E \).

Thm 13.10
Let \( \{F_k\} \) be a nested sequence of sets (i.e. \( F_k \subseteq F_{k+1} \) \( \forall k \in \mathbb{N} \)) which are all closed bounded nonempty sets of \( \mathbb{R}^n \). Then,
\[
F = \bigcap_{k=1}^{\infty} F_k
\]
is also closed bounded and nonempty.

Defn
A family \( \mathcal{U} \) of open sets is an open cover for a set \( E \subseteq S \) if each point in \( E \) belongs to at least one set in \( \mathcal{U} \) (i.e. \( E \subseteq \bigcup \mathcal{U} \)).

A subcover of \( \mathcal{U} \) is a subfamily of \( \mathcal{U} \) which also covers \( E \).

A cover or subcover is finite if only contains a finite number of sets, each of which may contain an infinite number of points.

A set \( E \) is compact if every open cover of \( E \) has a finite subcover of \( E \).

Proposition 13.13
Every n-cell in \( \mathbb{R}^n \) is compact.

Note: An n-cell \( F \) is an n-dimensional box \( F = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n] \).

In \( \mathbb{R}^2 \) this is just a rectangle.

Heine-Borel Theorem (13.12)
A set \( E \subseteq \mathbb{R}^n \) is compact if and only if it is closed and bounded.

Defn
A set \( E \) in \( \mathbb{R}^n \) is bounded if there exists an \( M > 0 \) where \( \max \{ |x_j| ; j = 1, \ldots, n \} \leq M \) \( \forall x = (x_1, x_2, \ldots, x_n) \in E \).