Math 25
Vogler

Subsequences

Defn
Given a sequence $\{s_n\}_{n=1}^{\infty}$ of real numbers and a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $n_k < n_{k+1}$ $\forall k \in \mathbb{N}$, the sequence $\{s_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{s_n\}_{n=1}^{\infty}$. If $\{s_{n_k}\}_{k=1}^{\infty}$ converges, then its limit is called a subsequential limit of $\{s_n\}_{n=1}^{\infty}$.

Notes: 1) The notation $\{\ \}_{n=1}^{\infty}$ is used to convey what is the index of the sequence, along with the range of the index. This notation is usually dropped when its clear what the index is and the range is $1 \to \infty$ (i.e. $\mathbb{N}$).
2) The subsequence index $\{n_k\}$ is always a strictly increasing (i.e. nondecreasing) sequence.
3) A subsequence is just a subset of the original sequence where you preserve the order of the terms.
4) $\{n_k\}$ can be viewed as a function on $\mathbb{N}$ and $\{s_{n_k}\}$ as functional composition. Refer to the book for more details.

Thm (11.2)
If $\{s_n\}$ converges, then all its subsequences converge to its limit.

Thm (11.3)
Every sequence has a monotone subsequence.

Cor (11.4)
Let $\{s_n\}$ be any sequence. There exists monotonic subsequence whose limit is $\limsup_{n\to\infty} s_n$, and there exists a monotonic subsequence whose limit is $\liminf_{n\to\infty} s_n$.

Bolzano-Weierstrass Theorem (11.5)
Every bounded sequence has a convergent subsequence.

Notes: 1) This is a very useful fact because it's not very easy to construct a convergent sequence with the properties you desire, but it's usually easier to construct a bounded sequence with the desired properties and this theorem gives you a convergent subsequence for 'free'.
2) This fact will get used alot in Math 125A.

Thm (11.7)
Let $\{s_n\}$ be a sequence and S be the set of subsequential limits. Then we have the following:
1) S is nonempty.
2) $\sup S = \limsup_{n\to\infty} s_n$ and $\inf S = \liminf_{n\to\infty} s_n$.
3) $\lim_{n\to\infty} s_n = L$ if and only if $S = \{L\}$ (i.e. S has one element, L)

Notes: a) Fact 2) is extremely useful in computing lim inf and lim sup of a sequence.
b) Fact 3) is useful in proving when $\{s_n\}$ does not converge.

Thm (11.8)
Let $\{s_n\}$ be a sequence and S be the set of subsequential limits. Assume $\{t_n\}$ is a sequence in $\mathbb{R} \cap S$ and $\lim_{n\to\infty} t_n = t$. Then $t \in S$.