Homework 1 Solutions

1.1) (i) This is true for $n = 1$, since $1^2 = 1 = \frac{1(2)(3)}{6}$.

(ii) Let $n \in \mathbb{N}$. Suppose that

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

is true. Then, we have

$$1^2 + 2^2 + ... + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = (n+1)\left(\frac{n(2n+1)}{6} + \frac{6(n+1)}{6}\right) \quad (1)$$

$$= (n+1)\left(\frac{2n^2+7n+6}{6}\right) = \frac{(n+1)(n+2)(2n+3)}{6}. \quad (2)$$

So the statement is true for $n+1$.

Therefore, by the Principle of Mathematical Induction, we conclude

$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{N}.$$

1.4) (a) Since the sum gives $1, 4, 9, 16$ for $n = 1, 2, 3, 4$, respectively. We propose the following formula

$$1 + 3 + 5 + ... + (2n-1) = n^2 \quad \forall n \in \mathbb{N}.$$

(b) (i) This is true for $n = 1$, since $1 = 1^2$.

(ii) Let $n \in \mathbb{N}$. Suppose that

$$1 + 3 + 5 + ... + (2n-1) = n^2$$

is true. Then, we have

$$1 + 3 + 5 + ... + (2n-1) + (2n+1) = n^2 + (2n+1) = (n+1)^2. \quad (3)$$

So the statement is true for $n+1$.

Therefore, by the Principle of Mathematical Induction, we conclude

$$1 + 3 + 5 + ... + (2n-1) = n^2 \quad \forall n \in \mathbb{N}.$$

1.7) (i) This is true for $n = 1$, since $7^1 - 6 - 1 = 0$ is divisible by 36.

(ii) Let $n \in \mathbb{N}$. Suppose that $7^n - 6n - 1$ is divisible by 36 is true. So, $\exists k \in \mathbb{Z}$ such that $7^n - 6n - 1 = 36k$. Then, we obtain

$$7^{n+1} - 6(n+1) - 1 = 7^{n+1} - 6n - 7 = 7(7^n - 6n - 1) + 36n = 7(36k) + 36n = 36(7k+n),$$

where $7k + n \in \mathbb{Z}$. So $7^{n+1} - 6(n+1) - 1$ is divisible by 36, and the statement is true for $n+1$.

Therefore, by the Principle of Mathematical Induction, we conclude $7^n - 6n - 1$ is divisible by 36 $\forall n \in \mathbb{N}$.

1.8) (b) (i) This is true for $n = 4$, since $4! = 24 > 16 = 4^2$.

(ii) Let $n \in \mathbb{N}$ with $n \geq 4$. Suppose that $n! > n^2$ is true. Then, we have

$$(n+1)! = (n+1)n! > (n+1)n^2$$

and

$$n \geq 4 \Rightarrow n^2 \geq 4n = n + 3n \geq n + 12 > n + 1.$$

Thus, we combine and obtain

$$(n+1)! > (n+1)n^2 > (n+1)(n+1) = (n+1)^2.$$

So the statement is true for $n+1$.

Therefore, by the Principle of Mathematical Induction, we conclude $n! > n^2$ $\forall n \in \mathbb{N}$ with $n \geq 4$.

1.9) (i) With some guess and check work, we have $2^n > n^2$ for $n = 0, 1$ or when $n \geq 5$. Thus, the statement we wish to prove is

$$2^n > n^2 \quad \forall n \in \mathbb{N} \text{ with } n \geq 5$$

(ii) Let $n \in \mathbb{N}$ with $n \geq 5$. Suppose that $2^n > n^2$ is true. Then, we have

$$2^{n+1} = 2(2^n) > 2n^2$$

and

$$n \geq 5 \Rightarrow 2n^2 = n^2 + n^2 \geq n^2 + 5n = n^2 + 2n + 3n > n^2 + 2n + 1 = (n+1)^2.$$

Thus, we combine and obtain

$$2^{n+1} > 2n^2 > (n+1)^2.$$

So the statement is true for $n+1$.

Therefore, by the Principle of Mathematical Induction, we conclude

$$2^n > n^2 \quad \forall n \in \mathbb{N} \text{ with } n \geq 5$$

Worksheet 1 Solutions

1) (i) This is true for $n = 1$, since $1 \cdot 2 = 2 = \frac{1(2)(3)}{3}$.

(ii) Let $n \in \mathbb{N}$. Suppose that

$$1 \cdot 2 + 2 \cdot 3 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

is true. Then, we have

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + ... + n(n+1) + (n+1)(n+2) &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \qquad (4) \\ &= (n+1)(n+2)\left(\frac{n}{3} + \frac{3}{3}\right) = \frac{(n+1)(n+2)(n+3)}{3}. \qquad (5) \end{aligned}$$

So the statement is true for $n + 1$.

Therefore, by the Principle of Mathematical Induction, we conclude

$$1 \cdot 2 + 2 \cdot 3 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3} \quad \forall n \in \mathbb{N}.$$

2) (i) This is true for $n = 1$, since $1 + x \geq 1 + x$.

(ii) Let $n \in \mathbb{N}$. Suppose that

$$(1+x)^n \geq 1 + nx$$

is true. Since $1 + x > 0$ and $nx^2 \geq 0$, we have

$$(1+x)^{n+1} = (1+x)(1+x)^n \geq (1+x)(1+nx) = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x. \qquad (6)$$

So the statement is true for $n + 1$.

Therefore, by the Principle of Mathematical Induction, we conclude

$$(1+x)^n \geq 1 + nx \quad \forall n \in \mathbb{N}.$$

(The following is a proof by contrapositive)

3) Suppose $m \in \mathbb{Z}$ and m is not even. So m must be odd, and $\exists k \in \mathbb{Z}$ such that $m = 2k + 1$. Thus, we have

$$m^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$$

with $(4k^3 + 6k^2 + 3k) \in \mathbb{Z}$, so m^3 is odd. Thus, m^3 is not even.

Therefore, if m^3 is even, then m is even.

4) Suppose $x \in \mathbb{I}$ and $r \in \mathbb{Q}$, but $x+r$ is not irrational. So $x+r \in \mathbb{Q}$. Then, we must have $x = (x+r)-r$ is rational (If you don't believe this, it is quite straightforward to show two rational numbers added together are rational), but x is rational. Contradiction!

Therefore, if x is irrational and r is rational, then $x + r$ is irrational.

5) Suppose instead that $\sqrt{5}$ is rational, so $\exists m, n \in \mathbb{Z}$ with $n \neq 0$ and m and n have no common factors such that

$$\sqrt{5} = \frac{m}{n} \Rightarrow 5 = \frac{m^2}{n^2} \Rightarrow m^2 = 5n^2. \tag{7}$$

Then, $5|m^2$. Since 5 is prime, we have $5|m$, so $\exists k \in \mathbb{Z}$ such that $m = 5k$. Plugging this into (7), we get

$$25k^2 = 5n^2 \Rightarrow 5k^2 = n^2.$$

Thus, $5|n^2$. Since 5 is prime, we have $5|n$. Thus, we showed $5|m$ and $5|n$, but we assumed m and n have no common factors. Contradiction!

Therefore, $\sqrt{5}$ must be irrational.