Homework 10 Solutions

10.2) Suppose \( \{s_n\} \) is a bounded non-increasing sequence. Let \( \epsilon > 0 \) be given. Define \( S = \{s_n : n \in \mathbb{N}\} \). Since \( S \) is bounded, \( \inf S \in \mathbb{R} \) exists by a Corollary of the Completeness Axiom. Let \( l = \inf S \). Since \( l < l + \epsilon \) and \( l = \inf S \), \( \exists x \in S \) such that \( x < l + \epsilon \), and \( \exists N \in \mathbb{N} \) such that \( s_N = x \). Then, we have

\[
l - \epsilon < l \leq s_n < l + \epsilon \quad \forall n > N,
\]

since the sequence is non-increasing. Choose this \( N \). Thus, \( \forall n > N \), then \( |s_n - l| < \epsilon \).

Therefore, we conclude \( \lim_{n \to \infty} s_n = l = \inf S \).

10.7) Suppose \( S \) is a bounded nonempty subset of \( \mathbb{R} \) and \( \sup S \notin S \). Define \( t = \sup S \). We are going to construct the sequence \( \{s_n\} \) through construction by induction as follows:

(i) Consider \( t - 1 < t \). Since \( t = \sup S \), \( \exists s_1 \in S \) such that \( t - 1 < s_1 < t \) because \( t \notin S \). So \( s_1 \) exists.

(ii) Let \( n \in \mathbb{N} \). Suppose that

\[
t - \frac{1}{j} < s_j < t \quad \text{and} \quad s_{j-1} \leq s_j \quad \text{for} \quad j = 1, 2, ..., n
\]

is true (Note: we do not really need this to construct the next sequence element, and it is here for completeness of the induction hypothesis). Consider \( m = \max \{t - \frac{1}{n+1}, s_n\} < t \). Since \( t = \sup S \), \( \exists s_{n+1} \in S \) such that \( m < s_{n+1} < t \) because \( t \notin S \). So \( s_{n+1} \) exists with the property

\[
t - \frac{1}{n+1} < s_{n+1} < t \quad \text{and} \quad s_n \leq s_{n+1}.
\]

Therefore, by the Principle of Complete Induction, we constructed a non-decreasing sequence \( \{s_n\} \) with the property

\[
t - \frac{1}{n} < s_n < t \quad \forall n \in \mathbb{N}.
\]

Since \( \lim_{n \to \infty} t - \frac{1}{n} = \lim_{n \to \infty} t = t \), we have \( \lim_{n \to \infty} s_n = t = \sup S \) by the Squeeze Theorem.

Therefore, we conclude if \( S \) is a bounded nonempty subset of \( \mathbb{R} \) and \( \sup S \notin S \), then there exists a non-decreasing sequence \( \{s_n\} \) with \( s_n \in S \) such that \( \lim_{n \to \infty} s_n = \sup S \).

10.9) Let \( s_1 = 1 \) and \( s_{n+1} = \frac{n}{n+1} s_n^2 \) for \( n \geq 1 \) (i.e. a recursion relation).

(a) \( s_1 = 1, \ s_2 = \frac{1}{2}, \ s_3 = \frac{1}{6}, \) and \( s_4 = \frac{1}{48} \).

(b) We can show that the sequence \( \{s_n\} \) is bounded between 0 and 1 (i.e. \( 0 < s_n \leq 1 \ \forall n \in \mathbb{N} \)) by using induction (optional homework problem). Also, notice we have

\[
s_{n+1} = \frac{n}{n+1} s_n^2 < s_n^2 \leq 1 \cdot s_n = s_n \quad \forall n \in \mathbb{N}.
\]
This gives us $s_{n+1} \leq s_n \ \forall n \in \mathbb{N}$, and the sequence is monotonically non-increasing. Since $\{s_n\}$ is a bounded monotone sequence, it must converge by Theorem 10.2.

(c) From part (b), we know the sequence converges. Let $s = \lim_{n \to \infty} s_n$. Then, from the recursion relation, we have

\[
\lim_{n \to \infty} s_{n+1} = \lim_{n \to \infty} \frac{n}{n + 1} s_n^2 \Rightarrow s = s^2 \Rightarrow s(s - 1) = 0 \Rightarrow s = 1 \text{ or } s = 0.
\]

Since $s_n \leq \frac{1}{2}$ for $n \geq 2$ and non-increasing, we must have $s \leq \frac{1}{2}$, then $s = 0$.

Therefore, we conclude $\lim_{n \to \infty} s_n = 0$. 
Worksheet 5 Solutions

1) (a) Suppose \( \lim_{n \to \infty} s_n = \infty \). Let \( \epsilon > 0 \) be given. Notice that

\[
\left| \frac{1}{s_n} - 0 \right| = \left| \frac{1}{s_n} \right| = \frac{1}{s_n}
\]

Since \( \lim_{n \to \infty} s_n = \infty \), \( \exists N \in \mathbb{N} \) such that \( \forall n > N \), then \( s_n > \frac{1}{\epsilon} \), which is equivalent to

\[
\forall n > N \Rightarrow \frac{1}{s_n} < \epsilon.
\]

Choose this \( N \). Thus, we have

\[
\forall n > N \Rightarrow \left| \frac{1}{s_n} - 0 \right| = \frac{1}{s_n} < \epsilon.
\]

Therefore, we conclude \( \lim_{n \to \infty} \frac{1}{s_n} = 0 \).

(b) Suppose \( \lim_{n \to \infty} \frac{1}{s_n} = 0 \) and \( s_n > 0 \ \forall n \in \mathbb{N} \). Let \( M > 0 \) be given. Since \( \lim_{n \to \infty} s_n = 0 \), \( \exists N \in \mathbb{N} \) such that \( \forall n > N \), then \( |s_n - 0| = s_n < \frac{1}{M} \), which is equivalent to

\[
\forall n > N \Rightarrow \frac{1}{s_n} > M.
\]

Choose this \( N \). Thus, we have

\[
\forall n > N \Rightarrow \left| \frac{1}{s_n} - 0 \right| = \frac{1}{s_n} > M.
\]

Therefore, we conclude \( \lim_{n \to \infty} \frac{1}{s_n} = \infty \).

2) Let \( s_1 = \sqrt{5} \) and \( s_{n+1} = \sqrt{5 + s_n} \) for \( n \geq 1 \) (i.e. a recursion relation).

(a) First, we show \( \{s_n\} \) is bounded between 0 and 3 with induction:

(i) This is true for \( n = 1 \), since \( 0 < \sqrt{5} < \sqrt{9} = 3 \).

(ii) Let \( n \in \mathbb{N} \). Suppose that \( 0 < s_n < 3 \) is true. Then, we have

\[
0 < s_{n+1} = \sqrt{5 + s_n} < \sqrt{5 + 3} < \sqrt{9} = 3
\]

So the statement is true for \( n + 1 \).

Thus, by the Principle of Mathematical Induction, we have \( 0 < s_n < 3 \ \forall n \in \mathbb{N} \), and the sequence is bounded.

Second, we show \( s_n \leq s_{n+1} \ \forall n \in \mathbb{N} \) (i.e. sequence is non-decreasing) by induction:
(i) This is true for $n = 1$, since $s_1 = \sqrt{5} < \sqrt{5 + \sqrt{5}} = s_2$.

(ii) Let $n \in \mathbb{N}$. Suppose that $s_n \leq s_{n+1}$ is true. Then, we have $5 + s_n \leq 5 + s_{n+1}$, so

$$s_{n+1} = \sqrt{5 + s_n} \leq \sqrt{5 + 5} = s_{n+2}.$$ 

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we have $s_n \leq s_{n+1} \ \forall n \in \mathbb{N}$, and the sequence is monotonically non-decreasing.

Therefore, $\{s_n\}$ is a bounded monotone sequence, and we conclude it must converge by Theorem 10.2.

(b) From part (a), we know the sequence converges. Let $s = \lim_{n \to \infty} s_n$. Then, from the recursion relation, we have

$$\lim_{n \to \infty} s_{n+1} = \lim_{n \to \infty} \sqrt{5 + s_n} \Rightarrow s = \sqrt{s + 5} \Rightarrow s^2 - s - 5 = 0 \Rightarrow s = \frac{1 \pm \sqrt{21}}{2}.$$ 

Since $s_n > 0 \ \forall n \in \mathbb{N}$ and the sequence is non-decreasing, we must have $s > 0$, then $s = \frac{1 + \sqrt{21}}{2}$.

Therefore, we conclude $\lim_{n \to \infty} s_n = \frac{1 + \sqrt{21}}{2}$.

3) Let $s_1 = 2$ and $s_{n+1} = \frac{1}{4} s_n + 15$ for $n \geq 1$ (i.e. a recursion relation).

(a) First, we show $\{s_n\}$ is bounded between 0 and 20 with induction:

(i) This is true for $n = 1$, since $0 < s_1 = 2 < 20$.

(ii) Let $n \in \mathbb{N}$. Suppose that $0 < s_n < 20$ is true. Then, we have

$$0 < s_{n+1} = \frac{1}{4} s_n + 15 < \frac{1}{4} (20) + 15 = 20.$$ 

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we have $0 < s_n < 20 \ \forall n \in \mathbb{N}$, and the sequence is bounded.

Second, we show $s_n \leq s_{n+1} \ \forall n \in \mathbb{N}$ (i.e. sequence is non-decreasing) by induction:

(i) This is true for $n = 1$, since $s_1 = 2 < \frac{1}{2} + 15 = s_2$.

(ii) Let $n \in \mathbb{N}$. Suppose that $s_n \leq s_{n+1}$ is true. Then, through algebraic manipulation we have

$$s_{n+1} = \frac{1}{4} s_n + 15 \leq \frac{1}{4} s_{n+1} + 15 = s_{n+2}.$$ 

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we have $s_n \leq s_{n+1} \ \forall n \in \mathbb{N}$, and the sequence is monotonically non-decreasing.

Therefore, $\{s_n\}$ is a bounded monotone sequence, and we conclude it must converge by Theorem 10.2.
(b) From part (a), we know the sequence converges. Let $s = \lim_{n \to \infty} s_n$. Then, from the recursion relation, we have

$$\lim_{n \to \infty} s_{n+1} = \lim_{n \to \infty} \frac{1}{4}s_n + 15 \Rightarrow s = \frac{1}{4}s + 15 \Rightarrow s = 20.$$ 

Therefore, we conclude $\lim_{n \to \infty} s_n = 20$. 