Homework 10 Solutions

10.2) Suppose $\{s_n\}$ is a bounded non-increasing sequence. Let $\epsilon > 0$ be given. Define $S = \{s_n : n \in \mathbb{N}\}$. Since S is bounded, inf $S \in \mathbb{R}$ exists by a Corollary of the Completeness Axiom. Let $l =$ inf S. Since $l < l + \epsilon$ and $l =$inf S, $\exists x \in S$ such that $x < l + \epsilon$, and $\exists N \in \mathbb{N}$ such that $s_N = x$. Then, we have

$$l - \epsilon < l \leq s_n \leq s_N < l + \epsilon \quad \forall n > N,$$

since the sequence is non-increasing. Choose this N. Thus, $\forall n > N$, then $|s_n - l| < \epsilon$.

Therefore, we conclude $\lim_{n\to\infty} s_n = l =$ inf S.

10.7) Suppose S is a bounded nonempty subset of $\mathbb{R}$ and sup $S \notin S$. Define $t =$ sup S. We are going to construct the sequence $\{s_n\}$ through construction by induction as follows:

(i) Consider $t - 1 < t$. Since $t =$ sup S, $\exists s_1 \in S$ such that $t - 1 < s_1 < t$ because $t \notin S$. So s_1 exists.

(ii) Let $n \in \mathbb{N}$. Suppose that

$$t - \frac{1}{j} < s_j < t \text{ and } s_{j-1} \leq s_j \text{ for } j = 1, 2, ..., n$$

is true (Note: we do not really need this to construct the next sequence element, and it is here for completeness of the induction hypothesis). Consider $m =$ max $\{t - \frac{1}{n+1}, s_n\} < t$. Since $t =$ sup S, $\exists s_{n+1} \in S$ such that $m < s_{n+1} < t$ because $t \notin S$. So s_{n+1} exists with the property

$$t - \frac{1}{n+1} < s_{n+1} < t \text{ and } s_n \leq s_{n+1}.$$

Therefore, by the Principle of Complete Induction, we constructed a non-decreasing sequence $\{s_n\}$ with the property

$$t - \frac{1}{n} < s_n < t \quad \forall n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} t - \frac{1}{n} = \lim_{n\to\infty} t = t$, we have $\lim_{n\to\infty} s_n = t =$ sup S by the Squeeze Theorem.

Therefore, we conclude if S is a bounded nonempty subset of $\mathbb{R}$ and sup $S \notin S$, then there exists a non-decreasing sequence $\{s_n\}$ with $s_n \in S$ such that $\lim_{n\to\infty} s_n =$ sup S.

10.9) Let $s_1 = 1$ and $s_{n+1} = \frac{n}{n+1} s_n^2$ for $n \geq 1$ (i.e. a recursion relation).

(a) $s_1 = 1$, $s_2 = \frac{1}{2}$, $s_3 = \frac{1}{6}$, and $s_4 = \frac{1}{48}$.

(b) We can show that the sequence $\{s_n\}$ is bounded between 0 and 1 (i.e. $0 < s_n \leq 1 \;\; \forall n \in \mathbb{N}$) by using induction (optional homework problem). Also, notice we have

$$s_{n+1} = \frac{n}{n+1} s_n^2 < s_n^2 \leq 1 \cdot s_n = s_n \quad \forall n \in \mathbb{N}.$$

This gives us $s_{n+1} \leq s_n \quad \forall n \in \mathbb{N}$, and the sequence is monotonically non-increasing. Since $\{s_n\}$ is a bounded monotone sequence, it must converge by Theorem 10.2.

(c) From part (b), we know the sequence converges. Let $s = \lim_{n\to\infty} s_n$. Then, from the recursion relation, we have

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{n}{n+1} s_n^2 \Rightarrow s = s^2 \Rightarrow s(s-1) = 0 \Rightarrow s = 1 \text{ or } s = 0.$$

Since $s_n \leq \frac{1}{2}$ for $n \geq 2$ and non-increasing, we must have $s \leq \frac{1}{2}$, then $s = 0$.

Therefore, we conclude $\lim_{n\to\infty} s_n = 0$

Worksheet 5 Solutions

1) (a) Suppose $\lim_{n\to\infty} s_n = \infty$. Let $\epsilon > 0$ be given. Notice that

$$\left|\frac{1}{s_n} - 0\right| = |\frac{1}{s_n}| = \frac{1}{s_n}$$

Since $\lim_{n\to\infty} s_n = \infty$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, then $s_n > \frac{1}{\epsilon}$, which is equivalent to

$$\forall n > N \Rightarrow \frac{1}{s_n} < \epsilon.$$

Choose this N. Thus, we have

$$\forall n > N \Rightarrow \left|\frac{1}{s_n} - 0\right| = \frac{1}{s_n} < \epsilon.$$

Therefore, we conclude $\lim_{n\to\infty} \frac{1}{s_n} = 0$.

(b) Suppose $\lim_{n\to\infty} \frac{1}{s_n} = 0$ and $s_n > 0\ \forall n \in \mathbb{N}$. Let $M > 0$ be given. Since $\lim_{n\to\infty} s_n = 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, then $|s_n - 0| = s_n < \frac{1}{M}$, which is equivalent to

$$\forall n > N \Rightarrow \frac{1}{s_n} > M.$$

Choose this N. Thus, we have

$$\forall n > N \Rightarrow \left|\frac{1}{s_n} - 0\right| = \frac{1}{s_n} > M.$$

Therefore, we conclude $\lim_{n\to\infty} \frac{1}{s_n} = \infty$.

2) Let $s_1 = \sqrt{5}$ and $s_{n+1} = \sqrt{5 + s_n}$ for $n \geq 1$ (i.e. a recursion relation).

(a) First, we show $\{s_n\}$ is bounded between 0 and 3 with induction:

(i) This is true for $n = 1$, since $0 < \sqrt{5} < \sqrt{9} = 3$.

(ii) Let $n \in \mathbb{N}$. Suppose that $0 < s_n < 3$ is true. Then, we have

$$0 < s_{n+1} = \sqrt{5 + s_n} < \sqrt{5 + 3} < \sqrt{9} = 3$$

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we have $0 < s_n < 3\ \ \forall n \in \mathbb{N}$, and the sequence is bounded.

Second, we show $s_n \leq s_{n+1}\ \forall n \in \mathbb{N}$ (i.e. sequence is non-decreasing) by induction:

(i) This is true for $n = 1$, since $s_1 = \sqrt{5} < \sqrt{5 + \sqrt{5}} = s_2$.

(ii) Let $n \in \mathbb{N}$. Suppose that $s_n \leq s_{n+1}$ is true. Then, we have $5 + s_n \leq 5 + s_{n+1}$, so

$$s_{n+1} = \sqrt{5 + s_n} \leq \sqrt{5 + s_{n+1}} = s_{n+2}.$$

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we have $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$, and the sequence is monotonically non-decreasing.

Therefore, $\{s_n\}$ is a bounded monotone sequence, and we conclude it must converge by Theorem 10.2.

(b) From part (a), we know the sequence converges. Let $s = \lim_{n\to\infty} s_n$. Then, from the recursion relation, we have

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \sqrt{5 + s_n} \Rightarrow s = \sqrt{s + 5} \Rightarrow s^2 - s - 5 = 0 \Rightarrow s = \frac{1 \pm \sqrt{21}}{2}.$$

Since $s_n > 0$ $\forall n \in \mathbb{N}$ and the sequence is non-decreasing, we must have $s > 0$, then $s = \dfrac{1 + \sqrt{21}}{2}$.

Therefore, we conclude $\lim_{n\to\infty} s_n = \dfrac{1 + \sqrt{21}}{2}$.

3) Let $s_1 = 2$ and $s_{n+1} = \frac{1}{4}s_n + 15$ for $n \geq 1$ (i.e. a recursion relation).

(a) First, we show $\{s_n\}$ is bounded between 0 and 20 with induction:

(i) This is true for $n = 1$, since $0 < s_1 = 2 < 20$.

(ii) Let $n \in \mathbb{N}$. Suppose that $0 < s_n < 20$ is true. Then, we have

$$0 < s_{n+1} = \frac{1}{4}s_n + 15 < \frac{1}{4}(20) + 15 = 20$$

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we have $0 < s_n < 20$ $\forall n \in \mathbb{N}$, and the sequence is bounded.

Second, we show $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$ (i.e. sequence is non-decreasing) by induction:

(i) This is true for $n = 1$, since $s_1 = 2 < \frac{1}{2} + 15 = s_2$.

(ii) Let $n \in \mathbb{N}$. Suppose that $s_n \leq s_{n+1}$ is true. Then, through algebraic manipulation we have

$$s_{n+1} = \frac{1}{4}s_n + 15 \leq \frac{1}{4}s_{n+1} + 15 = s_{n+2}.$$

So the statement is true for $n + 1$.

Thus, by the Principle of Mathematical Induction, we have $s_n \leq s_{n+1}$ $\forall n \in \mathbb{N}$, and the sequence is monotonically non-decreasing.

Therefore, $\{s_n\}$ is a bounded monotone sequence, and we conclude it must converge by Theorem 10.2.

(b) From part (a), we know the sequence converges. Let $s = \lim_{n\to\infty} s_n$. Then, from the recursion relation, we have

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \frac{1}{4}s_n + 15 \Rightarrow s = \frac{1}{4}s + 15 \Rightarrow s = 20.$$

Therefore, we conclude $\lim_{n\to\infty} s_n = 20$.