Homework 13 Solutions

For the next problem, we need the following fact which is the result of Exercise 5.4 (an optional homework problem):

**Proposition 1.** Let $S \subset \mathbb{R}$ be nonempty. Define $S^- := \{-s : s \in S\}$. Then $\inf S = -\sup S^-.$

11.8) Suppose we have a sequence $\{s_n\}$. We define the set of terms in the sequence as $E := \{s_n : n \in \mathbb{N}\}$. We consider two cases: (1) $E$ is bounded below or (2) $E$ is not bounded below.

For case (1), suppose $E$ is bounded below. For any $N \in \mathbb{N}$, define $E_N := \{s_n : n > N\}$ and $E_N^- := \{-s_n : n > N\}$. Then by definition, we have

$$\liminf_{n \to \infty} s_n = \lim_{N \to \infty} \inf E_N = \lim_{N \to \infty} -\sup E_N^-,$$

using Proposition 1. By definition again, we continue further obtain

$$\liminf_{n \to \infty} s_n := \lim_{N \to \infty} \inf E_N = \lim_{N \to \infty} -\sup E_N^- = -\lim_{N \to \infty} \sup E_N^- := -\limsup_{n \to \infty} -s_n.$$

Thus, for this case we have

$$\liminf_{n \to \infty} s_n = -\limsup_{n \to \infty} -s_n.$$

For case (1), suppose $E$ is not bounded below. Then, the corresponding sequence $\{s_n\}$ is not bounded below too, and consequently the sequence $\{-s_n\}$ is not bounded above. Hence, we have

$$\liminf_{n \to \infty} s_n = -\infty = -\limsup_{n \to \infty} -s_n.$$ 

Therefore, we conclude

$$\liminf_{n \to \infty} s_n = -\limsup_{n \to \infty} -s_n.$$

11.11) Suppose $S$ is a bounded nonempty set of $\mathbb{R}$. By the Completeness Axiom, $\sup S \in \mathbb{R}$ exists. Define $L := \sup S$. There are two cases to consider: (1) $L \in S$ or (2) $L \not\in S$.

For case (1), if $L \in S$ (i.e. $L$ is a maximum), we can easily build a monotone sequence that converges to $L$ by defining $s_n = L \forall n \in \mathbb{N}$, and we are done.

For case (2), suppose $L \not\in S$. We are going to construct a non-decreasing (or increasing) sequence $\{s_n\}$ of points in $S$ (i.e. $s_n \in S \forall n \in \mathbb{N}$) such that $\lim_{n \to \infty} s_n = L$ through construction by induction as follows:

(i) Consider $L - 1 < L$. Since $L = \sup S$, $\exists s_1 \in S$ such that $L - 1 < s_1 < L$ because $L \not\in S$. So $s_1$ exists.

(ii) Let $n \in \mathbb{N}$. Suppose that

$$L - \frac{1}{j} < s_j < L$$

and $s_{j-1} \leq s_j$ for $j = 1, 2, ..., n$
is true (Note: we do not really need this to construct the next sequence element, and it is here for completeness of the induction hypothesis). Consider \( m = \max \{L - \frac{1}{n+1}, s_n\} < L \). Since \( L = \sup S \), \( \exists s_{n+1} \in S \) such that \( m < s_{n+1} < L \) because \( L \notin S \). So \( s_{n+1} \) exists with the property

\[
L - \frac{1}{n+1} < s_{n+1} < L \quad \text{and} \quad s_n < s_{n+1}.
\]

Therefore, by the Principle of Complete Induction, we constructed a non-decreasing (or increasing) sequence \( \{s_n\} \) with the property

\[
L - \frac{1}{n} < s_n < L \quad \forall n \in \mathbb{N}.
\]

Since \( \lim_{n \to \infty} L - \frac{1}{n} = \lim_{n \to \infty} L = L \), we have \( \lim_{n \to \infty} s_n = t = L = \sup S \) by the Squeeze Theorem.

Therefore, we conclude if \( S \) is a bounded nonempty subset of \( \mathbb{R} \), then there exists a non-decreasing sequence \( \{s_n\} \) with \( s_n \in S \) \( \forall n \in \mathbb{N} \) such that \( \lim_{n \to \infty} s_n = \sup S \).
Worksheet 6 Solutions

For the next problem, we need the following inequality:

**Proposition 2.**

\[
\frac{1}{m^2} \leq \frac{1}{m(m-1)} = \frac{1}{m-1} - \frac{1}{m} \quad \text{for } m \geq 2.
\]

2) Let

\[ s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \quad \forall n \in \mathbb{N}. \]

First notice that the sequence \( \{s_n\} \) is clearly non-decreasing (or increasing). By repeated use of Proposition 2, we have the following inequality

\[
s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} \leq 1 + \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \ldots + \left( \frac{1}{n-1} - \frac{1}{n} \right) = 2 - \frac{1}{n} < 2 \quad \forall n \in \mathbb{N},
\]

where a huge amount of cancellation occurs because it's a telescoping sum. So, the sequence is also bounded.

Therefore, we conclude \( \{s_n\} \) converges by Theorem 10.2 (which some people call the Monotone Convergence Theorem).

Remark: This is a proof that the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges.

3) (a) Let

\[
s_n = \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \ldots + \frac{k(n)}{n!} \quad \text{with } k(n) = \frac{1}{3} \left( 4 \cos \left( \frac{(2n-3)\pi}{3} \right) + 1 \right) \quad \forall n \in \mathbb{N}.
\]

Since all but the last term cancels out, we obtain

\[
|s_{n+1} - s_n| = \left| \pm \frac{1}{(n+1)!} \right| = \frac{1}{(n+1)!} \leq \frac{1}{2n} \quad \forall n \in \mathbb{N},
\]

because \( (n+1)! = (n+1) \cdot n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 \geq 2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2 \cdot 1 = 2^n \). Therefore, we conclude that \( |s_{n+1} - s_n| \quad \forall n \in \mathbb{N} \).

(b) By Homework 10.6 (a), we conclude that \( \{s_n\} \) is a Cauchy sequence. Also, \( \{s_n\} \) converges by Theorem 10.11.