Homework 14 Solutions

12.4) Suppose $\{s_n\}$ and $\{t_n\}$ are bounded sequences. For every $N \in \mathbb{N}$, define $A_N := \{s_n + t_n : n > N\}$, $B_N := \{s_n : n > N\}$, and $C_N := \{t_n : n > N\}$. Since all the sequences are bounded, their sup's must exist as real numbers by the Completeness Axiom. Let $a_N = \sup A_N$, $b_N = \sup B_N$, and $c_N = \sup C_N$ for every $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$. If $n \geq N$, then $s_n + t_n \leq b_N + c_N$, so $b_N + c_N$ is an upper bound for A_N, and thus $a_N \leq b_N + c_N$. Since $N \in \mathbb{N}$ was arbitrary, we have $a_N \leq b_N + c_N$ $\forall N \in \mathbb{N}$. Taking the limit of both sides gives us

$$\lim_{N\to\infty} a_N \leq \lim_{N\to\infty} (b_N + c_N) = \lim_{N\to\infty} b_N + \lim_{N\to\infty} c_N.$$

Therefore, we conclude if $\{s_n\}$ and $\{t_n\}$ are bounded sequences, then

$$\limsup_{n\to\infty} (s_n + t_n) \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} t_n$$

by definition of lim sup.

12.5) Suppose $\{s_n\}$ and $\{t_n\}$ are bounded sequences. Using Homework 12.4 once (the inequality) and Homework 11.8a three times (first and last equality), we easily obtain

$$\liminf_{n\to\infty} (s_n + t_n) = -\limsup_{n\to\infty} (-s_n - t_n) \geq -\left(\limsup_{n\to\infty} -s_n + \limsup_{n\to\infty} -t_n\right)$$

$$= -\limsup_{n\to\infty} -s_n - \limsup_{n\to\infty} -t_n = \liminf_{n\to\infty} s_n + \liminf_{n\to\infty} t_n.$$

Therefore, we conclude if $\{s_n\}$ and $\{t_n\}$ are bounded sequences, then

$$\liminf_{n\to\infty} (s_n + t_n) \geq \liminf_{n\to\infty} s_n + \liminf_{n\to\infty} t_n.$$

12.6) Let $\{s_n\}$ be a bounded sequence and $k \geq 0$.

(a) For each $N \in \mathbb{N}$, define $E_N := \{s_n : n > N\}$ and $kE_N := \{ks_n : n > N\}$. Since both the sequences $\{s_n\}$ and $\{k \cdot s_n\}$ are bounded, their sup's must exist as real numbers by the Completeness Axiom. One can easily show (optional homework) that $\sup (kE_N) = k \sup E_N$ $\forall N \in \mathbb{N}$. Taking limits of both sides and using definition of lim sup, we obtain

$$\limsup_{n\to\infty} k \cdot s_n = \lim_{N\to\infty} \sup (kE_N) = k\left(\lim_{N\to\infty} \sup E_N\right) = k\left(\limsup_{n\to\infty} s_n\right).$$

Therefore, we conclude if $\{s_n\}$ be a bounded sequence and $k \geq 0$, then

$$\limsup_{n\to\infty} k \cdot s_n = k\left(\limsup_{n\to\infty} s_n\right).$$

(b) By Homework 11.8a and part (a) above, we easily obtain

$$\liminf_{n\to\infty} k \cdot s_n = -\limsup_{n\to\infty} -k \cdot s_n = -\limsup_{n\to\infty} k(-s_n)$$

$$= -k\left(\limsup_{n\to\infty}\ -s_n\right) = k\left(-\limsup_{n\to\infty}\ -s_n\right) = k\left(\liminf_{n\to\infty}\ s_n\right).$$

Therefore, we conclude if $\{s_n\}$ be a bounded sequence and $k \geq 0$, then

$$\liminf_{n\to\infty}\ k\cdot s_n = k\left(\liminf_{n\to\infty}\ s_n\right).$$

(c) If instead $k < 0$, then $-k > 0$, and using Homework 11.8 (parts b and c) we obtain the following:

$$\limsup_{n\to\infty}\ k\cdot s_n = \limsup_{n\to\infty}\ (-k)(-s_n) = -k\left(\limsup_{n\to\infty}\ -s_n\right) = k\left(-\limsup_{n\to\infty}\ -s_n\right) = k\left(\liminf_{n\to\infty}\ s_n\right),$$

and

$$\liminf_{n\to\infty}\ k\cdot s_n = \liminf_{n\to\infty}\ (-k)(-s_n) = -k\left(\liminf_{n\to\infty}\ -s_n\right) = k\left(-\liminf_{n\to\infty}\ -s_n\right) = k\left(\limsup_{n\to\infty}\ s_n\right).$$

12.8) Suppose $\{s_n\}$ and $\{t_n\}$ are bounded sequences of nonnegative numbers. For every $N \in \mathbb{N}$, define $A_N := \{s_n t_n : n > N\}$, $B_N := \{s_n : n > N\}$, and $C_N := \{t_n : n > N\}$. Since all the sequences are bounded, their sup's must exist as real numbers by the Completeness Axiom. Let $a_N =\ \sup A_N$, $b_N =\ \sup B_N$, and $c_N =\ \sup C_N$ for every $N \in \mathbb{N}$. Fix $N \in \mathbb{N}$. If $n \geq N$, then $s_n t_n \leq b_N c_N$ since $s_n \leq b_N$, $t_n \leq c_N$ and $s_n, t_n \geq 0\ \forall n \in \mathbb{N}$, so $b_N c_N$ is an upper bound for A_N, and thus $a_N \leq b_N c_N$. Since $N \in \mathbb{N}$ was arbitrary, we have $a_N \leq b_N c_N\ \forall N \in \mathbb{N}$. Taking the limit of both sides gives us

$$\lim_{N\to\infty}\ a_N \leq \lim_{N\to\infty}\ b_N c_N = \lim_{N\to\infty}\ b_N \cdot \lim_{N\to\infty}\ c_N.$$

Therefore, we conclude if $\{s_n\}$ and $\{t_n\}$ are bounded sequences of nonnegative numbers, then

$$\limsup_{n\to\infty}\ (s_n t_n) \leq \limsup_{n\to\infty}\ s_n \cdot \limsup_{n\to\infty}\ t_n$$

by definition of lim sup.

12.11) The solution for this will be posted later. This problem (or the equivalent one presented in class) WILL NOT appear on Exam 2.