Homework 16 Solutions

14.10) Consider the series $\sum_{n=1}^{\infty} \frac{2^n}{n^{2+(-1)^n}}$. The sequence of terms for this series can be formulated as

$$a_n = \begin{cases} \frac{2^n}{n} & \text{if } n \text{ is odd} \\ \frac{2^n}{n^3} & \text{if } n \text{ is even} \end{cases}$$

Notice we have $\frac{2^n}{n^3} \leq \frac{2^n}{n}$ $\forall n \in \mathbb{N}$. Also, the ratios for this sequence can be formulated as follows

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \frac{2n}{(n+1)^3} & \text{if } n \text{ is odd} \\ \frac{2n^3}{n+1} & \text{if } n \text{ is even} \end{cases}$$

Notice that $\frac{2n}{(n+1)^3} \leq \frac{2n^3}{n+1}$ $\forall n \in \mathbb{N}$. Thus, we have the following calculations

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n^3}{n+1} = \infty > 1 \quad \text{and} \quad \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n}{(n+1)^3} = 0 < 1.$$

So the Ratio Test is inconclusive. But, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n}} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n}} = \frac{2}{1} = 2 > 1.$$

Therefore, we conclude the series diverges by the Root Test.

14.3) (b) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Notice that the partial fraction decomposition is

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \quad \forall n \in \mathbb{N}.$$

Then, the sequence of partial sums $\{s_n\}$ for the series becomes

$$s_n = a_1 + a_2 + ... + a_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Since this is a telescoping sum, we have $s_n = 1 - \frac{1}{n+1}$ $\forall n \in \mathbb{N}$. Therefore, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1,$$

and we conclude $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

Worksheet 7 Solutions

5) Let $N \in \mathbb{N}$. Also, let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with $a_n = b_n \ \forall n \geq N$. For $N = 1$, the statement is obvious. When $N > 1$, we have

$$\sum_{n=1}^{\infty} a_n \text{ converges } \Leftrightarrow \sum_{n=N}^{\infty} a_n \text{ converges } \Leftrightarrow \sum_{n=N}^{\infty} b_n \text{ converges } \Leftrightarrow \sum_{n=1}^{\infty} b_n \text{ converges },$$

by repeated use of Problem 4 on this worksheet (for $m = N-1$). Therefore, we conclude $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Note: We could also use the Cauchy Criteria to prove this since $|s_n - s_m| = |t_n - t_m|$ for $n > m \geq N-1$, where $\{s_n\}$ and $\{t_n\}$ are the sequences of partial sums.

6) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Suppose $a_n \leq b_n \ \forall n \in \mathbb{N}$ and there exists $m \in \mathbb{N}$ where $a_m < b_m$. Let $\{s_n\}$ and $\{t_n\}$ be the sequence of partial sums for $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, respectively. Then, we clearly have

$$s_m = a_1 + a_2 + ... + a_{m-1} + a_m < b_1 + b_2 + ... + b_{m-1} + b_m = t_m,$$

Plus, we know $\sum_{n=1}^{\infty} a_n = s_m + \sum_{n=m+1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n = t_m + \sum_{n=m+1}^{\infty} b_n$ by Problem 4 on this worksheet. Moreover, since $a_n \leq b_n \ \forall n \in \mathbb{N}$, we get $\sum_{n=m+1}^{\infty} a_n \leq \sum_{n=m+1}^{\infty} b_n$. Putting all these equalities and inequalities together, we obtain the following

$$\sum_{n=1}^{\infty} a_n = s_m + \sum_{n=m+1}^{\infty} a_n < t_m + \sum_{n=m+1}^{\infty} b_n = \sum_{n=1}^{\infty} b_n.$$

Therefore, we conclude if $a_n \leq b_n \ \forall n \in \mathbb{N}$ and there exists $m \in \mathbb{N}$ where $a_m < b_m$, then $\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} b_n$.

7) (a) Let $S_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + ... + \dfrac{1}{n}$.
(i) This is true for $m = 1$, since $S_{2^1} = 1 + \frac{1}{2}$.

(ii) Let $m \in \mathbb{N}$. Suppose that $S_{2^m} \geq 1 + \dfrac{m}{2}$ is true. Then, we have

$$S_{2^{m+1}} = S_{2^m} + \left(\frac{1}{2^m+1} + \frac{1}{2^m+2} + ... + \frac{1}{2^{m+1}}\right) \geq 1 + \frac{m}{2} + \left(\frac{1}{2^m+1} + \frac{1}{2^m+2} + ... + \frac{1}{2^{m+1}}\right)$$

Since $\frac{1}{2^{m+1}} \le \frac{1}{2^{m+1}-1} \le ... \le \frac{1}{2^m+2} \le \frac{1}{2^m+1}$, we get

$$S_{2^{m+1}} \ge 1 + \frac{m}{2} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + ... + \frac{1}{2^{m+1}}\right) = 1 + \frac{m}{2} + 2^m \frac{1}{2^{m+1}} = 1 + \frac{m+1}{2},$$

where we had 2^m terms of $\frac{1}{2^{m+1}}$. Thus, the statement is true for $m+1$.

Therefore, by the Principle of Mathematical Induction, we conclude $S_{2^m} \ge 1 + \frac{m}{2}$ $\forall m \in \mathbb{N}$.

(b) Let $M > 0$ be given. Without loss of generality, assume $M > 1$. By Archimedian Property, there exists $m \in \mathbb{N}$ such that $m \ge 2(M-1)$. Choose $N = 2^m$. Thus, $\forall n > N$, then

$$s_n \ge s_N = s_{2^m} \ge 1 + \frac{m}{2} \ge 1 + (M-1) = M,$$

since $\{s_n\}$ is increasing and using part (a).

Therefore, we conclude $\lim_{n\to\infty} s_n = \infty$ and the harmonic series diverges.

8) Suppose that the harmonic series converges, and let $\sum_{n=1}^{\infty} \frac{1}{n} = s$. Then, by definition

$$s = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + ...$$

Using Problem 6 on this worksheet along with the Comparison Test, we have

$$s > \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + ...$$

Using Problem 3 on this handout with the fact we have a convergent series, we get

$$s = \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + ... = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = s.$$

Thus, we obtain $s > s$, which is clearly a Contradiction! Therefore, the harmonic series must converge.

9) Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series with nonnegative terms. Since $\sum_{n=1}^{\infty} b_n$ converges, we have $\lim_{n\to\infty} b_n = 0$ by Nth Term Test. By choosing $\epsilon = 1 > 0$, since $\{b_n\}$ converges to 0, $\exists N \in \mathbb{N}$ such that $\forall n > N$, then $|b_n - 0| < 1$. So we have $b_n < 1$ $\forall n > N$.

Now, looking the product sequence $\{a_n b_n\}$, we notice $a_n b_n \le a_n(1) = a_n$ $\forall n > N$. Therefore, the series $\sum_{n=1}^{\infty} a_n b_n$ converges the the Comparison Test since $\sum_{n=1}^{\infty} a_n$ converges.