Homework 17 Solutions

15.3) This is an optional homework assignment.

15.4) (a) Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$. Notice, we have

$$\frac{1}{n \ln n} \leq \frac{1}{\sqrt{n}\ln n} \text{ for } n \geq 2$$

since $\sqrt{n} < n \ \forall n \in \mathbb{N}$. Since the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by Homework 15.3, the original series diverges by the Comparison Test.

(b) Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n}$. Notice, we have

$$\frac{1}{n} \leq \frac{\ln n}{n} \text{ for } n \geq 3.$$

Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by P-series Test ($p = 1 \leq 1$), the original series diverges by the Comparison Test.

(d) Consider the series $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$. Notice, we have

$$\frac{\ln n}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}} \text{ for } n \geq 2$$

since $\ln n < \sqrt{n} \ \forall n \in \mathbb{N}$. Since the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by P-series Test ($p = \frac{3}{2} > 1$), the original series converges by the Comparison Test.

Worksheet 9 Solutions

1) (a) True. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with all positive terms and $\sum_{n=1}^{\infty} a_n$ diverges. Since the terms a_n and b_n are all positive, we have $0 < a_n < a_n + b_n$ $\forall n \in \mathbb{N}$. Since the series $\sum_{n=1}^{\infty} a_n$ diverges , the series $\sum_{n=1}^{\infty} a_n + b_n$ diverges by the Comparison Test.

(b) False. For a counterexample, consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges (P-series Test with $p = 1 \leq 1$) and the series $\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{n}$. The second series diverges because $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (P-series Test with $p = 2 > 1$) and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (P-series Test with $p = 1 \leq 1$). Combining the two series through addition gives us

$$\sum_{n=1}^{\infty} \frac{1}{n} + \left(\frac{1}{n^2} - \frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which is a convergent series (P-series Test with $p = 2 > 1$).

2) (a) There are two cases to consider: 1) $x \geq 0$ or 2) $x < 0$.
For case 1), suppose $x \geq 0$, then $0 \leq x + |x| \leq 2|x|$ since $x + |x| = 2x = 2|x|$.
For case 2), suppose $x < 0$, then $0 \leq x + |x| \leq 2|x|$ since $x + |x| = x - x = 0$.
Either case, we conclude $0 \leq x + |x| \leq 2|x| \quad \forall x \in \mathbb{R}$.

(b) Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. Then, the series $\sum_{n=1}^{\infty} 2|a_n|$ converges by Homework 14.5b. By (a), we have $0 \leq a_n + |a_n| \leq 2|a_n| \quad \forall n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} 2|a_n|$ converges, the series $\sum_{n=1}^{\infty} a_n + |a_n|$ converges by the Comparison Test.

(c) Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. Then, the series $\sum_{n=1}^{\infty} -|a_n|$ converges by Homework 14.5b. By (b), we also have the series $\sum_{n=1}^{\infty} a_n + |a_n|$ converges. Thus, the resulting series $\sum_{n=1}^{\infty} (a_n + |a_n|) - |a_n| = \sum_{n=1}^{\infty} a_n$ converges by Homework 14.5a.

Therefore, we conclude if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges as well.

4) Consider the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\frac{1}{2}(3+(-1)^{n+1})}{\frac{1}{4}(2n+1+(-1)^{n+1})}$. Writing out the series, we get

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\frac{1}{2}(3+(-1)^{n+1})}{\frac{1}{4}(2n+1+(-1)^{n+1})}=\frac{2}{1}-\frac{1}{1}+\frac{2}{2}-\frac{1}{2}+\frac{2}{3}-\frac{1}{3}+...+\frac{2}{n}-\frac{1}{n}+\frac{2}{n+1}-\frac{1}{n+1}+...$$

Notice that

$$\frac{1}{n}<\frac{2}{n+1}\text{ for }n>1,$$

so the sequence of terms $\{a_n\}$ are not eventually decreasing. Therefore, we conclude that the Alternating Series Test does not apply here.

Looking at the sequence of partial sums $\{s_n\}$, we see a pattern with the subsequence of even terms. In particular, let $\{s_{2k}\}$ be a subsequence to $\{s_n\}$. Then, we have

$$s_{2k}=(2-1)+\left(1-\frac{1}{2}\right)+\left(\frac{2}{3}-\frac{1}{3}\right)+...+\left(\frac{2}{2k}-\frac{1}{2k}\right)=1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2k}$$

The corresponding series to the subsequence of partial sums $\{s_{2k}\}$ is $\sum_{k=1}^{\infty}\frac{1}{2k}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}$ which is the divergent harmonic series. Then, $\lim_{k\to\infty} s_{2k}=\infty$, and consequently, $\lim_{n\to\infty} s_n=\infty$. Therefore, we conclude the original series diverges.