Homework 18 Solutions

For the next problem, we need the following fact (an optional homework problem) :

Proposition 1. *If the series* $\sum a_n$ *with the sequence of terms* $\{a_n\}$ *non-increasing (or decreasing), then* $a_n \geq 0 \ \ \forall n \in \mathbb{N}$.

15.7) Suppose the series $\sum a_n$ converges where the sequence of terms $\{a_n\}$ is non-increasing (or decreasing). Let $\epsilon > 0$ be given. Since $\sum a_n$ converges, it satisfies the Cauchy Criterion. Then, $\exists N_0 \in \mathbb{R}$ such that

$$\forall n \geq m > N_0 \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \frac{\epsilon}{2}.$$

Let $N_* = \lceil N_0 \rceil + 2$, so that $N_* + 1 > N_0$. Hence, we obtain the following

$$(n - N_*)a_n \leq |a_{N_*+1} + ... + a_n| < \frac{\epsilon}{2} \ \ \forall n > N_*$$

since $\{a_n\}$ is non-increasing (or decreasing) and $a_n \geq 0 \ \forall n \in \mathbb{N}$ by Proposition 1. Rewriting this inequality, we get

$$2(n - N_*)a_n < \epsilon \ \ \forall n > N_*.$$

Notice if $n > 2N_*$, then $2(n - N_*) = n + (n - 2N_*) > n$. So choose $N = 2N_*$. Thus, $\forall n > N$, then $|na_n - 0| = na_n < 2(n - N_*)a_n < \epsilon$ since $a_n \geq 0 \ \ \forall n \in \mathbb{N}$ by Proposition 1.

Therefore, we conclude if the series $\sum a_n$ converges where the sequence of terms $\{a_n\}$ is non-increasing (or decreasing), then $\lim_{n\to\infty} na_n = 0$.

(b) Notice the converse of (a) is

Proposition 2. *If* $\lim_{n\to\infty} na_n \neq 0$*, then the series* $\sum a_n$ *diverges or the sequence of terms* $\{a_n\}$ *is non-decreasing (or increasing).*

Consider the harmonic series $\sum \frac{1}{n}$, so the sequence of terms is $a_n = \frac{1}{n} \ \ \forall n \in \mathbb{N}$. Clearly,

$$\lim_{n\to\infty} na_n = \lim_{n\to\infty} 1 = 1 \neq 0.$$

Also, $\{a_n\}$ is non-increasing (or decreasing). Therefore, by (converse of) (a), we conclude $\sum \frac{1}{n}$ diverges.

Worksheet 8 Solutions

5) Let $\sum_{n=1}^{\infty} a_n$ be series whose terms are positive and decreasing, and

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = 2a_2 + 4a_4 + 8a_8 + ... + 2^n a_{2^n} + ...$$

($\Rightarrow$) Suppose $\sum_{n=1}^{\infty} a_n$ converges. Clearly, $\sum_{n=2}^{\infty} a_n$ converges as well. Writing out this series and grouping gives us

$$\sum_{n=2}^{\infty} a_n = a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + ...$$

Since $\{a_n\}$ is decreasing, the previous terms in each group are smaller than the last term in the group, and we obtain the inequality

$$\frac{1}{2}\sum_{n=1}^{\infty} 2^n a_{2^n} = a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + ... \leq \sum_{n=2}^{\infty} a_n.$$

Hence, the series $\frac{1}{2}\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges by the Comparison Test. Therefore, we conclude $\sum_{n=1}^{\infty} 2^n a_{2^n}$ must converge by Homework 14.5b.

($\Leftarrow$) Suppose $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges. Writing out this series and grouping gives us

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (a_8 + ... + a_8) + ...$$

Since $\{a_n\}$ is decreasing, any entry in the group ends up be bigger than any subsequent element in the sequence $\{a_n\}$, and we obtain the inequality

$$\sum_{n=2}^{\infty} a_n = (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + ... \leq \sum_{n=1}^{\infty} 2^n a_{2^n}.$$

Hence, the series $\sum_{n=2}^{\infty} a_n$ converges by the Comparison Test. Therefore, we conclude $\sum_{n=1}^{\infty} a_n$ must also converge.

6) Consider the series $\sum_{n=2}^{\infty} 2^n \frac{1}{2^n(\ln 2^n)^2}$ (in order to use the result of Problem 5). Notice that

$$\sum_{n=2}^{\infty} 2^n \frac{1}{2^n(\ln 2^n)^2} = \sum_{n=2}^{\infty} \frac{1}{(n\ln 2)^2} = \sum_{n=2}^{\infty} \frac{1}{(\ln 2)^2}\frac{1}{n^2},$$

and this series converges by the P-series Test ($p = 2 > 1$) and Homework 14.5b. By Problem 5 on this worksheet, we conclude $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges.