Homework 18 Solutions

For the next problem, we need the following fact (an optional homework problem):

Proposition 1. If the series $\sum a_n$ with the sequence of terms $\{a_n\}$ non-increasing (or decreasing), then $a_n \geq 0 \quad \forall n \in \mathbb{N}$.

15.7) Suppose the series $\sum a_n$ converges where the sequence of terms $\{a_n\}$ is non-increasing (or decreasing). Let $\epsilon > 0$ be given. Since $\sum a_n$ converges, it satisfies the Cauchy Criterion. Then, $\exists N_0 \in \mathbb{R}$ such that

$$\forall n \geq m > N_0 \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \frac{\epsilon}{2}.$$ 

Let $N_* = \lceil N_0 \rceil + 2$, so that $N_* + 1 > N_0$. Hence, we obtain the following

$$(n - N_*)a_n \leq |a_{N_*+1} + \ldots + a_n| < \frac{\epsilon}{2} \quad \forall n > N_*$$

since $\{a_n\}$ is non-increasing (or decreasing) and $a_n \geq 0 \quad \forall n \in \mathbb{N}$ by Proposition 1. Rewriting this inequality, we get

$$2(n - N_*)a_n < \epsilon \quad \forall n > N_*.$$ 

Notice if $n > 2N_*$, then $2(n - N_*) = n + (n - 2N_*) > n$. So choose $N = 2N_*$. Thus, $\forall n > N$, then $|na_n - 0| = na_n < 2(n - N_*)a_n < \epsilon$ since $a_n \geq 0 \quad \forall n \in \mathbb{N}$ by Proposition 1.

Therefore, we conclude if the series $\sum a_n$ converges where the sequence of terms $\{a_n\}$ is non-increasing (or decreasing), then $\lim_{n \to \infty} na_n = 0$.

(b) Notice the converse of (a) is

Proposition 2. If $\lim_{n \to \infty} na_n \neq 0$, then the series $\sum a_n$ diverges or the sequence of terms $\{a_n\}$ is non-decreasing (or increasing).

Consider the harmonic series $\sum \frac{1}{n}$, so the sequence of terms is $a_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$. Clearly,

$$\lim_{n \to \infty} na_n = \lim_{n \to \infty} 1 = 1 \neq 0.$$ 

Also, $\{a_n\}$ is non-increasing (or decreasing). Therefore, by (converse of) (a), we conclude $\sum \frac{1}{n}$ diverges.
Worksheet 8 Solutions

5) Let \( \sum_{n=1}^{\infty} a_n \) be a series whose terms are positive and decreasing, and

\[
\sum_{n=1}^{\infty} 2^n a_{2^n} = 2a_2 + 4a_4 + 8a_8 + \ldots + 2^n a_{2^n} + \ldots
\]

(⇒) Suppose \( \sum_{n=1}^{\infty} a_n \) converges. Clearly, \( \sum_{n=2}^{\infty} a_n \) converges as well. Writing out this series and grouping gives us

\[
\sum_{n=2}^{\infty} a_n = a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) + \ldots
\]

Since \( \{a_n\} \) is decreasing, the previous terms in each group are smaller than the last term in the group, and we obtain the inequality

\[
\frac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2^n} = a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) + \ldots \leq \sum_{n=2}^{\infty} a_n.
\]

Hence, the series \( \frac{1}{2} \sum_{n=1}^{\infty} 2^n a_{2^n} \) converges by the Comparison Test. Therefore, we conclude \( \sum_{n=1}^{\infty} 2^n a_{2^n} \) must converge by Homework 14.5b.

(⇐) Suppose \( \sum_{n=1}^{\infty} 2^n a_{2^n} \) converges. Writing out this series and grouping gives us

\[
\sum_{n=1}^{\infty} 2^n a_{2^n} = (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) + (a_8 + \ldots + a_8) + \ldots
\]

Since \( \{a_n\} \) is decreasing, any entry in the group ends up being bigger than any subsequent element in the sequence \( \{a_n\} \), and we obtain the inequality

\[
\sum_{n=2}^{\infty} a_n = (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + \ldots \leq \sum_{n=1}^{\infty} 2^n a_{2^n}.
\]

Hence, the series \( \sum_{n=2}^{\infty} a_n \) converges by the Comparison Test. Therefore, we conclude \( \sum_{n=1}^{\infty} a_n \) must also converge.

6) Consider the series \( \sum_{n=2}^{\infty} \frac{1}{2^n (\ln 2^n)^2} \) (in order to use the result of Problem 5). Notice that

\[
\sum_{n=2}^{\infty} \frac{1}{2^n (\ln 2^n)^2} = \sum_{n=2}^{\infty} \frac{1}{(n \ln 2)^2} = \sum_{n=2}^{\infty} \frac{1}{(\ln 2)^2 n^2},
\]

and this series converges by the P-series Test \( (p = 2 > 1) \) and Homework 14.5b. By Problem 5 on this worksheet, we conclude \( \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2} \) converges.