Homework 19 Solutions

13.1) For the following problem, let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$.

Consider the metric $d_1(\vec{x}, \vec{y}) := \max_{j=1,...,n}\{|x_j - y_j|\}$.

(i) We have $d_1(\vec{x}, \vec{x}) := \max_{j=1,...,n}\{|x_j - x_j|\} = 0$. Also, if $\vec{x} \neq \vec{y}$, then there exists j such that $x_j \neq y_j$, and consequently, $|x_j - y_j| > 0$ because $(\mathbb{R}, |\cdot|)$ is a metric space. Therefore, $d_1(\vec{x}, \vec{y}) := \max_{j=1,...,n}\{|x_j - y_j|\} > 0$.

(ii) It's clear that

$$d_1(\vec{x}, \vec{y}) := \max_{j=1,...,n}\{|x_j - y_j|\} = \max_{j=1,...,n}\{|y_j - x_j|\} =: d_1(\vec{y}, \vec{x}).$$

(iii) Fix $j = 1, ..., n$. Then, we have

$$|x_j - z_j| = |x_j - y_j + y_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$$

by the Triangle Inequality in $\mathbb{R}$. Taking the maximum over $j = 1, ..., n$ of both sides gives us

$$d_1(\vec{x}, \vec{z}) := \max_{j=1,...,n}\{|x_j - z_j|\} \leq \max_{j=1,...,n}\{|x_j - y_j| + |y_j - z_j|\} \tag{1}$$

$$\leq \max_{j=1,...,n}\{|x_j - y_j|\} + \max_{j=1,...,n}\{|y_j - z_j|\} =: d_1(\vec{x}, \vec{y}) + d_1(\vec{y}, \vec{z}), \tag{2}$$

where we used the definition of the metric on both sides of the inequality.

Therefore, we conclude $d_1(\vec{x}, \vec{y})$ is a metric, and the pair $(\mathbb{R}^n, d_1)$ is a metric space.

Now, consider the metric $d_2(\vec{x}, \vec{y}) := \sum_{j=1}^{n} |x_j - y_j|$.

(i) We have $d_2(\vec{x}, \vec{x}) := \sum_{j=1}^{n} |x_j - x_j| = 0$. Also, if $\vec{x} \neq \vec{y}$, then there exists j such that $x_j \neq y_j$, and consequently, $|x_j - y_j| > 0$ because $(\mathbb{R}, |\cdot|)$ is a metric space. Therefore, $d_2(\vec{x}, \vec{y}) := \sum_{j=1}^{n} |x_j - y_j| > 0$.

(ii) It's clear that

$$d_2(\vec{x},\vec{y}) := \sum_{j=1}^{n} |x_j - y_j| = \sum_{j=1}^{n} |y_j - x_j| =: d_2(\vec{y},\vec{x}).$$

(iii) Fix $j = 1, ..., n$. Then, we have

$$|x_j - z_j| = |x_j - y_j + y_j - z_j| \le |x_j - y_j| + |y_j - z_j|$$

by the Triangle Inequality in $\mathbb{R}$. Taking the sum over $j = 1, ..., n$ of both sides gives us

$$d_2(\vec{x},\vec{z}) := \sum_{j=1}^{n} |x_j - z_j| \le \sum_{j=1}^{n} (|x_j - y_j| + |y_j - z_j|) \tag{3}$$

$$= \sum_{j=1}^{n} |x_j - y_j| + \sum_{j=1}^{n} |y_j - z_j| =: d_2(\vec{x},\vec{y}) + d_2(\vec{y},\vec{z}), \tag{4}$$

where we used the definition of the metric on both sides of the inequality and the fact that this is a finite sum to 'distribute' the summation.

Therefore, we conclude $d_2(\vec{x},\vec{y})$ is a metric, and the pair $(\mathbb{R}^n, d_2)$ is a metric space.

(b) Before we start this problem we need a couple of facts, the proof of the first is an optional homework problem

Proposition 1. *Let $\vec{x},\vec{y} \in \mathbb{R}^n$. For any $j = 1, ..., n$, the following inequality holds*

$$|x_j - y_j| \le d_1(\vec{x},\vec{y}) \le d_2(\vec{x},\vec{y}) \le n \max_{j=1,\dots,n} \{|x_j - y_j|\}$$

where the metrics d_1 and d_2 are defined above.

Proposition 2. *Let $\{\vec{x}_k\}$ be a sequence is $\mathbb{R}^n$. If each coordinate sequence $\{x_{k,j}\}$ $(j = 1, ...n)$ converges in the metric space $(\mathbb{R}, |\cdot|)$, then the sequence $\{\vec{x}_k\}$ converges in both $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_2)$, where the metrics d_1 and d_2 are defined above..*

Proof of Proposition 2: Let $\{\vec{x}_k\}$ be a sequence is $\mathbb{R}^n$. Suppose each coordinate sequence $\{x_{k,j}\}$ converges to $\{x_j\}$ for $j = 1, ...n$. Consider the point

$\vec{x} := (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ (which is the point we will converge to). Let $\epsilon > 0$ be given. For each $j = 1, ...n$, since $\lim_{k\to\infty} x_{k,j} = x_j$, $\exists N_j \in \mathbb{R}$ such that

$$\forall k > N_j \Rightarrow |x_{k,j} - x_j| < \frac{\epsilon}{n}.$$

Choose $N := \max\{N_1, ..., N_n\}$. Thus, using Proposition 1 we have

$$\forall k > N \Rightarrow d_1(\vec{x}_k, \vec{x}) \leq d_2(\vec{x}_k, \vec{x}) \leq n \max_{j=1,\dots,n} \{|x_j - y_j|\} < n\frac{\epsilon}{n} = \epsilon.$$

Therefore, we conclude that if each coordinate sequence $\{x_{k,j}\}$ $(j = 1, ...n)$ converges, then the sequence $\{\vec{x}_k\}$ converges in both $(\mathbb{R}^n, d_1)$ and $(\mathbb{R}^n, d_2)$.

Proof of 13.1b: Consider the metric space $(\mathbb{R}^n, d_1)$ where the metric is defined above. Suppose the sequence $\{\vec{x}_k\}$ is Cauchy in this metric space. Let $\epsilon > 0$ be given. Since $\{\vec{x}_k\}$ is Cauchy, $\exists N \in \mathbb{R}$ such that

$$\forall k, m > N \Rightarrow d_1(\vec{x}_k, \vec{x}_m) < \epsilon. \tag{5}$$

Let $j = 1, ..., n$ be fixed, and consider the coordinate sequence $\{x_{k,j}\}$. Choose N given above. Thus, using Proposition 1 and (5) we have

$$\forall k, m > N \Rightarrow |x_{k,j} - x_{m,j}| < d_1(\vec{x}_k, \vec{x}_m) < \epsilon.$$

Hence, the coordinate sequence $\{x_{k,j}\}$ is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$ and is also a convergent sequence since $(\mathbb{R}, |\cdot|)$ is complete by Theorem 10.11. Let $\lim_{k\to\infty} x_{k,j} = x_j \in \mathbb{R}$.

Since $j = 1, ..., n$ was arbitrary, all coordinate sequences $\{x_{k,j}\}$ converges to some $x_j \in \mathbb{R}$. By Proposition 2, the sequence $\{\vec{x}_k\}$ converges in $(\mathbb{R}^n, d_1)$. Since $\{\vec{x}_k\}$ was an arbitrary Cauchy sequence, we have if $\{\vec{x}_k\}$ is Cauchy in this metric space, then it converges as well. Therefore, we conclude the metric space $(\mathbb{R}^n, d_1)$ is complete

Note: To show $(\mathbb{R}^n, d_2)$ is complete, the same exact proof applies where you replace d_1 with d_2.

13.2) (a) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Consider the metric space $(\mathbb{R}^n, d)$ with

$$d(\vec{x}, \vec{y}) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

We focus our attention on the first inequality. Fix $j = 1, ..., n$. Clearly, we have

$$|x_j - y_j| = \sqrt{(x_j - y_j)^2} \leq \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} =: d(\vec{x}, \vec{y})$$

Since $j = 1, ..., n$ was arbitrary, we conclude

$$|x_j - y_j| \leq d(\vec{x}, \vec{y}) \;\; \forall j = 1, ..., n,$$

proving the first inequality.

Now we prove the second inequality. For $j = 1, ..., n$, we have

$$(x_j - y_j) \leq \max_{i=1,...,n} \{|x_i - y_i|\} \Rightarrow (x_j - y_j)^2 \leq \max_{i=1,...,n} \{|x_i - y_i|^2\}.$$

Summing over all $j = 1, ..., n$, we obtain

$$\sum_{j=1}^{n} (x_j - y_j)^2 \leq n \max_{i=1,...,n} \{|x_i - y_i|^2\}$$

Finally, taking the square root of both side of the equation (this can be done since the square root function is increasing and 'preserves' inequalities), we get

$$\sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} \leq \sqrt{n \max_{i=1,...,n} \{|x_i - y_i|^2\}} = \sqrt{n} \max_{i=1,...,n} \{|x_i - y_i|\},$$

and we conclude

$$d(\vec{x}, \vec{y}) \leq \sqrt{n} \max_{i=1,...,n} \{|x_i - y_i|\},$$

proving the second inequality.

(b) ($\Rightarrow$) Suppose the sequence $\{\vec{x}_k\}$ converges to $\vec{x}$ in $\mathbb{R}^n$. Let $\epsilon > 0$ be given. Since $\lim_{k \to \infty} \vec{x}_k = \vec{x}$, $\exists N \in \mathbb{R}$ such that

$$\forall k > N \Rightarrow d(\vec{x}_k, \vec{x}) < \epsilon. \tag{6}$$

Fix $j = 1, ..., n$ and consider the coordinate sequence $\{x_{k,j}\}$. Choose this N given above. Then, by (6) and Homework 13.2a we have

$$\forall k > N \Rightarrow |x_{k,j} - x_j| \leq d(\vec{x}_k, \vec{x}) < \epsilon.$$

Hence, this coordinate sequence converges to $x_j \in \mathbb{R}$. Since $j = 1, ..., n$ was arbitrary, each coordinate sequence of $\{\vec{x}_k\}$ converges in $\mathbb{R}$. Therefore, we conclude, if the sequence $\{\vec{x}_k\}$ converges in $\mathbb{R}^n$, then each $j = 1, ..., n$ coordinate sequence $\{x_{k,j}\}$ converges in $\mathbb{R}$.

($\Leftarrow$) Let $\{\vec{x}_k\}$ be a sequence is $\mathbb{R}^n$. Suppose each coordinate sequence $\{x_{k,j}\}$ converges to $\{x_j\} \in \mathbb{R}$ for $j = 1, ...n$. Consider the point $\vec{x} := (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ (which is the point we will converge to). Let $\epsilon > 0$ be given. For each $j = 1, ...n$, since $\lim_{k\to\infty} x_{k,j} = x_j$, $\exists N_j \in \mathbb{R}$ such that

$$\forall k > N_j \Rightarrow |x_{k,j} - x_j| < \frac{\epsilon}{\sqrt{n}}.$$

Choose $N := \max\{N_1, ..., N_n\}$. Thus, using Homework 13.2a we have

$$\forall k > N \Rightarrow d(\vec{x}_k, \vec{x}) \leq \sqrt{n} \max_{j=1,\ldots,n} \{|x_j - y_j|\} < \sqrt{n}\frac{\epsilon}{\sqrt{n}} = \epsilon.$$

Therefore, we conclude that if each coordinate sequence $\{x_{k,j}\}$ $(j = 1, ...n)$ converges in $\mathbb{R}$, then the sequence $\{\vec{x}_k\}$ converges in $\mathbb{R}^n$.

13.3) (a) Let $B = \{\{x_k\} : \{x_k\} \text{ is bounded}\}$ (note that $\vec{x} \in B$ can be viewed as a vector with an infinite number of terms, i.e $\vec{x} := (x_1, x_2, ..., x_k, ...)$), and define the metric

$$d(\vec{x}, \vec{y}) := \sup_{j\in\mathbb{N}}\{|x_j - y_j|\}.$$

Let $\vec{x}, \vec{y}, \vec{z} \in B$.

(i) We have $d(\vec{x}, \vec{x}) := \sup_{j\in\mathbb{N}}\{|x_j - x_j|\} = 0$. Also, if $\vec{x} \neq \vec{y}$, then there exists $j \in \mathbb{N}$ such that $x_j \neq y_j$, and consequently, $|x_j - y_j| > 0$ because $(\mathbb{R}, |\cdot|)$ is a metric space. Therefore, $d(\vec{x}, \vec{y}) := \sup_{j\in\mathbb{N}}\{|x_j - y_j|\} > 0$.

(ii) It's clear that

$$d(\vec{x}, \vec{y}) := \sup_{j\in\mathbb{N}}\{|x_j - y_j|\} = \sup_{j\in\mathbb{N}}\{|y_j - x_j|\} =: d(\vec{y}, \vec{x}).$$

(iii) Fix $j \in \mathbb{N}$. Then, we have

$$|x_j - z_j| = |x_j - y_j + y_j - z_j| \leq |x_j - y_j| + |y_j - z_j|$$

by the Triangle Inequality in $\mathbb{R}$. Hence, we see

$$|x_j - z_j| \leq |x_j - y_j| + |y_j - z_j| \quad \forall j \in \mathbb{N}$$

Taking the supremum over $j \in \mathbb{N}$ of both sides gives us

$$d(\vec{x}, \vec{z}) := \sup_{j \in \mathbb{N}}\{|x_j - z_j|\} \leq \sup_{j \in \mathbb{N}}\{|x_j - y_j| + |y_j - z_j|\} \tag{7}$$

$$\leq \sup_{j \in \mathbb{N}}\{|x_j - y_j|\} + \sup_{j \in \mathbb{N}}\{|y_j - z_j|\} =: d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z}), \tag{8}$$

where we used the definition of the metric on both sides of the inequality.

Therefore, we conclude $d(\vec{x}, \vec{y})$ is a metric, and the pair (B, d) is a metric space.