Homework 2 Solutions

2.2) (a) Let $x = \sqrt[3]{2}$, then x solves the equation $x^3 - 2 = 0$. By the Rational Zero's Theorem, since solutions must be of the form $\frac{p}{q}$ where $p| - 3$ and $q|1$, the only rational solutions to this equation are $\pm 1, \pm 2$. Clearly, none of these solve the equation. Since $x = \sqrt[3]{2}$ solves the equation, $\sqrt[3]{2}$ cannot be rational. Therefore, $\sqrt[3]{2}$ is irrational.

(b) Let $x = \sqrt[7]{5}$, then x solves the equation $x^7 - 5 = 0$. By the Rational Zero's Theorem, since solutions must be of the form $\frac{p}{q}$ where $p| - 5$ and $q|1$, the only rational solutions to this equation are $\pm 1, \pm 5$. Clearly, none of these solve the equation. Since $x = \sqrt[7]{5}$ solves the equation, $\sqrt[7]{5}$ cannot be rational. Therefore, $\sqrt[7]{5}$ is irrational.

(c) Let $x = \sqrt[4]{13}$, then x solves the equation $x^4 - 13 = 0$. By the Rational Zero's Theorem, since solutions must be of the form $\frac{p}{q}$ where $p| - 13$ and $q|1$, the only rational solutions to this equation are $\pm 1, \pm 13$. Clearly, none of these solve the equation. Since $x = \sqrt[4]{13}$ solves the equation, $\sqrt[4]{13}$ cannot be rational. Therefore, $\sqrt[4]{13}$ is irrational.

2.3) Let $x = \sqrt{2 + \sqrt{2}}$, then we have

$$x = \sqrt{2 + \sqrt{2}} \Leftrightarrow x^2 = 2 + \sqrt{2} \Leftrightarrow (x^2 - 2)^2 = 2 \Leftrightarrow x^4 - 4x^2 + 2 = 0.$$

So x solves the equation $x^4 - 4x^2 + 2 = 0$. By the Rational Zero's Theorem, since solutions must be of the form $\frac{p}{q}$ where $p|2$ and $q|1$, the only rational solutions to this equation are $\pm 1, \pm 2$. Clearly, none of these solve the equation. Since $x = \sqrt{2 + \sqrt{2}}$ solves the equation, $\sqrt{2 + \sqrt{2}}$ cannot be rational. Therefore, $\sqrt{2 + \sqrt{2}}$ is irrational.

2.4) Let $x = \sqrt[3]{5 - \sqrt{3}}$, then we have

$$x = \sqrt[3]{5 - \sqrt{3}} \Leftrightarrow x^3 = 5 - \sqrt{3} \Leftrightarrow (x^3 - 5)^2 = 3 \Leftrightarrow x^6 - 10x^3 + 22 = 0.$$

So x solves the equation $x^6 - 10x^3 + 22 = 0$. By the Rational Zero's Theorem, since solutions must be of the form $\frac{p}{q}$ where $p|22$ and $q|1$, the only rational solutions to this equation are $\pm 1, \pm 2, \pm 11, \pm 22$. Notice we must have $x > 0$ since $5 > \sqrt{3}$ and $x < 2$ since $x < \sqrt[3]{8} = 2$, so the only rational option left for x to be is 1, but clearly, 1 does not solve the equation. Since $x = \sqrt[3]{5 - \sqrt{3}}$ solves the equation, $\sqrt[3]{5 - \sqrt{3}}$ cannot be rational. Therefore, $\sqrt[3]{5 - \sqrt{3}}$ is irrational.

Worksheet 2 Solutions

6) Suppose $a, b, c \in \mathbb{Z}$ with $a^2 + b^2 = c^2$, but a an b are both odd. Then, $\exists k, m \in \mathbb{Z}$ such that $a = 2k+1$ and $b = 2m + 1$. So, we obtain

$$c^2 = a^2 + b^2 = (2k+1)^2 + (2m+1)^2 = (4k^2+4k+1)+(4m^2+4m+1) = 2(2k^2+2k+2m^2+2m+1), \quad (1)$$

with $2k^2 + 2k + 2m^2 + 2m + 1 \in \mathbb{Z}$. Thus, c^2 is even. So c is even, and $\exists n \in \mathbb{Z}$ such that $c = 2n$. Plugging this into (1), we get

$$4n^2 = 2(2k^2 + 2k + 2m^2 + 2m + 1) \Rightarrow 2n^2 = 2(k^2 + k + m^2 + m) + 1.$$

So, we have an odd number equaling an even number. Contradiction!

Therfore, if $a, b, c \in \mathbb{Z}$ with $a^2 + b^2 = c^2$, then either a or b is even.

7) Let $n \in \mathbb{Z}$ with $n > 1$. Define $T = \{m \in \mathbb{N} : m > 1 \text{ and } m|n\}$. Then, T is nonempty since $n \in T$, so by the Well Ordering Principle, T has a least element b. If b is prime, then we are done.

Now, assume b is not prime. Then $\exists a \in \mathbb{Z}$ with $1 < a < b$ and $a|b$. Since $a|b$ and $b|n$, we have $a|n$. Thus, $a \in T$ and $a < b$, but b was the least element in T. Contradiction! Thus, b must be prime.

Therefore, since $n > 1$ was arbitrary, every $n > 1$ has a prime factor.

8) This if false! If $a = 2$ and $b = 3$, then $6|ab$, but 6 does not divide either a or b.