## Homework 20 Solutions

Before we start the next problem, we need a few facts. This first fact is almost a direct consequence of definitions and is an optional homework assignment

**Proposition 1.** *Let $(S, d)$ be a metric space with $E \subseteq S$. Then, $E^\circ$ is open and $E^-$ is closed. Moreover, we have $E^\circ \subseteq E \subseteq E^-$.*

**Proposition 2.** *Let $(S, d)$ be a metric space with $E \subseteq S$. If $x \in E^-$, then $\forall r > 0, \exists x_* \in \mathbb{E}$ such that $x_* \in B_r(x)$.*

Proof of Proposition 2: Let $E \subseteq S$. Suppose $x \in E^-$, but $\exists r > 0, \forall x_* \in E$ we have $x_* \notin B_r(x)$. We will use this $r > 0$. Since $B_r(x)$ is open and it shares no elements with $E$, $B_r(x)^C$ is a closed set with $E \subseteq B_r(x)^C$ and $x \notin B_r(x)^C$, but $x \in E^-$ so it must be in all closed sets containing $E$. Contradiction!

Therefore, we conclude if $x \in E^-$, then $\forall r > 0, \exists x_* \in \mathbb{E}$ such that $x_* \in B_r(x)$.

**Proposition 3.** *Let $(S, d)$ be a metric space with $E \subseteq S$. Then, $(E^\circ)^C = (E^C)^-$.*

Proof of Proposition 3: Let $E \subseteq S$. By Proposition 1, we have

$$E^\circ \subseteq E \Rightarrow E^C \subseteq (E^\circ)^C.$$

Suppose $x \in (E^C)^-$. Since $x \in (E^C)^-$, it must be in every closed set containing $E^C$. In particular, since $E^C \subseteq (E^\circ)^C$ and $(E^\circ)^C$ is closed (a compliment of an open set $E^\circ$), we must have $x \in (E^\circ)^C$. Thus, since $x \in (E^C)^-$ was arbitrary, we have $(E^C)^- \subseteq (E^\circ)^C$.

Now, suppose $x \in (E^\circ)^C$, but $x \notin (E^C)^-$. Then, there exists a closed set $U \subseteq S$ such that $E^C \subseteq U$, but $x \notin U$. So $x \in U^C \subseteq (E^C)^C = E$. Thus, by definition $x$ is an interior point of $E$ since $U^C$ is open. Hence, we have $x \in E^\circ$, but $x \in (E^\circ)^C$ by assumption. Contradiction!

Consequently, if $x \in (E^\circ)^C$, then $x \in (E^C)^-$, and we have $(E^\circ)^C \subseteq (E^C)^-$.

Therefore, we have $(E^C)^- \subseteq (E^\circ)^C$ and $(E^\circ)^C \subseteq (E^C)^-$, and we conclude $(E^\circ)^C = (E^C)^-$

13.6) (a) ($\Rightarrow$) Suppose $E$ is closed. Consider $C$ the collection of closed sets containing E. Since $E$ is closed and $E \subseteq E$, we have $E \in C$. Let $x \in E^-$, so we have $x \in U\ \forall U \in C$. Since $E \in C$, $x \in E$. Thus, $E^- \subseteq E$, and we also have $E \subseteq E^-$ by Proposition 1. Therefore, we conclude $E = E^-$.

($\Leftarrow$) Suppose $E = E^-$. Since $E^-$ is closed by Proposition 1, $(E^-)^C = E^C$ is open. Therefore, we conclude $E$ is closed by definition.

(b) ($\Rightarrow$) Suppose $E$ is closed, but there exists a sequence $\{s_n\}$ in $E$ (i.e. $s_n \in E\ \forall n \in \mathbb{N}$) that converges to $s \notin E$. So $s \in E^C$. Since $E$ is closed, $E^C$ is open and $s \in E^C$ is an interior point. Then, $\exists r > 0$ such that $B_r(s) \subseteq E^C$. Since $\lim\limits_{n\to\infty} s_n = s$, $\exists N \in \mathbb{R}$ such that $\forall n > N$, then $d(s_n, s) < r$. In particular, let $n = \lceil N \rceil + 1 > N$, so $d(s_n, s) < r$ which implies $s_n \in B_r(s)$ and $s_n \in E^C$. But $s_n \in E\ \forall n \in N$. Contradiction!

Therefore, we conclude if $E$ is closed then it contains the limit of every convergent sequence in $E$.

($\Leftarrow$) Suppose for every sequence $\{s_n\}$ in $E$ that converges to $s$, we have $s \in E$. Let $x \in E^-$. Fix $n \in \mathbb{N}$ and consider $r = \frac{1}{n} > 0$. By Proposition 2, $\exists s \in E$ such that $s \in B_{\frac{1}{n}}(x)$. Denote this element as $x_n = s$. Since $n \in \mathbb{N}$ was arbitrary, we have a sequence $\{x_n\}$ with the property $x_n \in B_{\frac{1}{n}}(x)\ \forall n \in \mathbb{N}$. Moreover, this implies $d(x_n, x) < \frac{1}{n}\ \ \forall n \in \mathbb{N}$. Hence, this sequence converges to $x$ (optional homework assignment). By assumption, $x \in E$. Since $x \in E^-$ was arbitrary, we have $E^- \subseteq E$, and we also have $E \subseteq E^-$ by Proposition 1. Therefore, we know $E = E^-$, and we conclude $E$ is closed by part (a).

Note: One could have used construction by induction to create the sequence $\{x_n\}$, but the induction step becomes trivial in this case. So a short-hand version of it was used here, along with the next problem.

(c) ($\Rightarrow$) Suppose $x \in E^-$. Fix $n \in \mathbb{N}$ and consider $r = \frac{1}{n} > 0$. By Proposition 2, $\exists s \in E$ such that $s \in B_{\frac{1}{n}}(x)$. Denote this element as $x_n = s$. Since $n \in \mathbb{N}$ was arbitrary, we have a sequence $\{x_n\}$ with the property $x_n \in B_{\frac{1}{n}}(x)\ \forall n \in \mathbb{N}$. Moreover, this implies $d(x_n, x) < \frac{1}{n}\ \ \forall n \in \mathbb{N}$. Hence, this sequence converges to $x$ (optional homework assignment).

Therefore, we conclude if $x \in E^-$, then it is the limit of some sequence of points in $E$.

($\Leftarrow$) Suppose there exists a sequence $\{x_n\}$ in $E$ (i.e. $x_n \in E\ \forall n \in \mathbb{N}$) that converges to $x$. Since $E \subseteq E^-$ from Proposition 1, the sequence $\{x_n\}$ is in $E^-$ as well. Because $E^-$ is closed, it contains the limit of every convergent sequence from part (b). In particular, we have $x \in E^-$.

Therefore, we conclude if there exists a sequence $\{x_n\}$ in $E$ that converges to $x$, then $x \in E^-$.

(d) The handout has a mistake about this fact. Here's the revised version:

**Proposition 4.** *$x \in \delta E$ if and only if $x \in E^-$ and $x \in (E^C)^-$.*

Proof: ($\Rightarrow$) Suppose $x \in \delta E$, then by definition $x \in E^- \setminus E^\circ$, and obviously, $x \in E^-$. Since $x \notin E^\circ$, we know $x \in (E^\circ)^C = (E^C)^-$ by Proposition 3. Thus, we conclude if $x \in \delta E$, then $x \in E^-$ and $x \in (E^C)^-$.

($\Leftarrow$) Suppose $x \in E^-$ and $x \in (E^C)^-$. Because $(E^C)^- = (E^\circ)^C$ by Proposition 3, we get $x \notin E^\circ$. We now have $x \in \delta E := E^- \setminus E^\circ$ since $x \in E^-$ and $x \notin E^\circ$.

Therefore, we conclude if $x \in E^-$ and $x \in (E^C)^-$, then $x \in \delta E$.

13.9) Before we start this problem, the best way to find the closure of the set $E$ is to notice two things:

1) Proposition 1 tells us that $E \subseteq E^-$, so we know the original set will always be in it's closure.

2) In light of Proposition 13.9c, which was proven above, we just need to look for limit points outside of the set $E$ which can be obtained with sequences in $E$.

With this in mind, we will always have $E^- = E \cup \{\text{limit points outside of } E\}$.

(a) Let $A := \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $A^- = A \cup \{0\}$.

(b) Let $B := \mathbb{Q}$. Because every irrational number in $\mathbb{I}$ can be the limit of a sequence of rational numbers (Practice Exam 1 Problem 8), we have $B^- = B \cup \mathbb{I} = \mathbb{Q} \cup \mathbb{I} = \mathbb{R}$.

(c) Let $C := \{r \in \mathbb{Q} : r^2 < 2\}$. Using similar logic as in part (b), we have $C^- = C \cup \{x \in \mathbb{I} : x^2 \leq 2\} = (-\infty, \sqrt{2}]$.

13.10) (a) Let $A := \{\frac{1}{n} : n \in \mathbb{N}\}$. Pick some element in $p \in A$. Let $r > 0$ be given. Notice, that $p < p + r$ so by the Denseness of the Irrationals (Homework 4.12), $\exists x \in \mathbb{I}$ such that $p < x < p + r$. Clearly, $x \in B_r(p)$ and $x \notin A$ since the set $A$ only contains rational numbers. Since $r > 0$ was arbitrary, there is not neighborhood about $p$ which is a subset of $A$, and $p$ is not an interior point of $A$. Because $p \in A$ was arbitrary, $p \notin A^\circ \;\; \forall p \in A$. Therefore, we conclude $A^\circ = \emptyset$.

(b) Let $B := \mathbb{Q}$. Pick some element in $p \in B$. Let $r > 0$ be given. Notice, that $p < p + r$ so by the Denseness of the Irrationals (Homework 4.12), $\exists x \in \mathbb{I}$ such that $p < x < p + r$. Clearly, $x \in B_r(p)$ and $x \notin B$ since the set $B$ only contains rational numbers. Since $r > 0$ was arbitrary, there is not neighborhood about $p$ which is a subset of $B$, and $p$ is not an interior point of $B$. Because $p \in B$ was arbitrary, $p \notin B^\circ \;\; \forall p \in B$. Therefore, we conclude $B^\circ = \emptyset$.