Homework 21 Solutions

13.11) Refer to the practice final exam solutions, problem 13.

13.12) Let (S, d) be a metric space.

(a) Suppose F is compact and $E \subseteq F$ is closed. Let $\mathcal{C}$ be an open cover of E. Because E is closed, E^C is open. Then, the family of sets $\mathcal{C}^* := \{U : U \in \mathcal{C}\} \cup \{E^C\}$ is an open cover of F. Since F is compact, there exists a finite subcover $\mathcal{C}_S^*$ that covers F, and define a subcover of E by $\mathcal{C}_S := \mathcal{C}_S^* \setminus \{E^C\}$ which covers E because $E \subseteq F$ and clearly E^C doesn't contain any points of E. Also, $\mathcal{C}_S$ is finite because it is a subset of a finite set (of sets). Since the cover $\mathcal{C}$ was arbitrary, we conclude E is compact.

(b) This will be an optional homework problem

13.13) Let E be a compact nonempty subset of $\mathbb{R}$. Since E is compact, it is closed and bounded by the Heine-Boral Thoerem, and $\sup E$ must exist in $\mathbb{R}$ by the Completeness Axiom. Because E is bounded and nonempty, there exists a non-decreasing (or increasing) sequence $\{s_n\}$ which converges to $\sup E$ by Homework 10.7. By Proposition 10.9b, the limit point, $\sup E$, must be in E since E is closed. Therefore, we conclude $\sup E \in E$.

Note: A similar argument can be used to prove $\inf E \in E$, or you could use the above proof with the fact that $\inf E = \sup E_-$ where $E_- := \{-x : x \in E\}$ (see Homework 4.9). Either way, this is an optional homework assignment.

13.15) This will be an optional homework assignment, which will not appear on the final.