Thus, we conclude $\frac{|a|}{|b|} = 1$ which implies $a = -(a)$, then we can obtain $ab = -(ab)$. Combining this with part (iii) of Theorem 3.1, the following occurs

$$(-a)(-b) = -(a(-b)) = -((-b)a) = -(-ba) = -(ab) = ab,$$

where we used the commutative law (M2) twice. Therefore, we conclude $(-a)(-b) = ab$.

(b) Suppose $ab = bc$ and $c \neq 0$. Then by M4 $\exists c^{-1} \in \mathbb{R}$ so we can multiply to get $(ac)c^{-1} = (bc)c^{-1}$. So $a(cc^{-1}) = b(cc^{-1})$ by M1, and this becomes $a1 = b1$ by the inverse property M4. Thus, we conclude $a = b$ by M3.

3.4) (a) By (iv) in Theorem 3.2, we have $0 \leq 1^2 = 1$, and since $0 \neq 1$ by M3, we conclude $0 < 1$.

(b) Suppose $0 < a < b$. Using (vi) of Theorem 3.2 twice, we have $a^{-1} > 0$ and $b^{-1} > 0$. Since $a < b$, we have $a^{-1}a < a^{-1}b$ by O5 and $1 < a^{-1}b$ by M4. Then $1(b^{-1}) < (a^{-1}b)b^{-1}$ by O5, so we have $b^{-1} < a^{-1}(bb^{-1})$ by M3 and M1. Finally, we can obtain $b^{-1} < a^{-1} \cdot 1 = a^{-1}$ by M4 and M3. Therefore, we conclude $0 < b^{-1} < a^{-1}$.

3.5) (b) Suppose $a, b \in \mathbb{R}$. We will consider the only two cases: (i) $|a| - |b| \geq 0$ and (ii) $|a| - |b| < 0$.

Case (i): Suppose $|a| - |b| \geq 0$. Then, we must have $|a| - |b| = |a| - |b|$ by definition of absolute value. By the Triangle Inequality, we get $|a| = |a - b + b| \leq |a - b| + |b|$. So, with subtraction by $|b|$, we obtain

$$|a| - |b| = |a| - |b| \leq (|a - b| + |b|) - |b| = |a - b|.$$ 

Thus, $|a| - |b| \leq |a - b|$.

Case (ii): Suppose $|a| - |b| < 0$. Then, we must have $|a| - |b| = -(|a| - |b|) = |b| - |a|$ by definition of absolute value. By the Triangle Inequality, we get $|b| = |(b - a) + a| \leq |b - a| + |a|$. So, with subtraction by $|a|$, we obtain

$$|a| - |b| = |b| - |a| \leq (|b - a| + |a|) - |a| = |b - a| = |a - b|.$$ 

Thus, we conclude $|a| - |b| \leq |a - b|$.

Since $a$ and $b$ were arbitrary, both cases give us $|a| - |b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

3.6) (b) (i) This is true for $n = 1$, since $|a_1| \leq |a_1|$ by equality.
(ii) Let $n \in \mathbb{N}$. Suppose that
\[
|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n| \tag{1}
\]
is true. Then, by the Triangle Inequality, we obtain
\[
|a_1 + a_2 + \ldots + a_n + a_{n+1}| = |(a_1 + a_2 + \ldots + a_n) + a_{n+1}| \leq |a_1 + a_2 + \ldots + a_n| + |a_{n+1}|
\]
So, using the induction hypothesis (1), we conclude
\[
|a_1 + a_2 + \ldots + a_n + a_{n+1}| \leq |a_1| + |a_2| + \ldots + |a_n| + |a_{n+1}|,
\]
and the statement is true for $n + 1$.

Therefore, by the Principle of Mathematical Induction, we conclude
\[
|a_1 + a_2 + \ldots + a_n| \leq |a_1| + |a_2| + \ldots + |a_n| \quad \forall n \in \mathbb{N}
\]
Worksheet 2 Solutions

1) Suppose $a \leq b$ and $c \leq d$. Since $a \leq b$, we have $a + c \leq b + c$ by O4. Similarly, since $c \leq d$, we obtain $b + c \leq b + d$ with O4 and A2. Using both of these inequalities along with O3, we conclude $a + c \leq b + d$.

2) Suppose $0 \leq a \leq b$ and $0 \leq c \leq d$. Since $a \leq b$ and $0 \leq c$, we get $ac \leq bc$ by O5. Similarly, since $c \leq d$ and $0 \leq b$, we obtain $bc \leq bd$ with O5 and M2. Combining these inequalities with O3, we conclude $ac \leq bd$.

3) Suppose $x > 0$, $y > 0$, and $x^2 < y^2$, but $x \geq y$. Since $x > 0$ and $x \geq y$, we have $x^2 \geq xy$ by O5 and M2. Similarly, since $y > 0$ and $x \leq y$, we get $xy \geq y^2$ with O5. Combining these inequalities with O3, we conclude $x^2 \geq y^2$. Contradiction!

Therefore, if $x > 0$, $y > 0$, and $x^2 < y^2$, then $x < y$.

4) Suppose $0 < x < y$.
   (i) This is true for $n = 1$, since $x < y$ by assumption.

   (ii) Let $n \in \mathbb{N}$. Suppose that $x^n < y^n$ is true. Since $x > 0$, multiplying both sides by $x$ yields $x^{n+1} < y^n$ by O5 and M2. Since $y^n > 0$, through repeated use of (iii) of Theorem 3.2, and $x < y$, we have $xy^n < y^{n+1}$ by O5. Combining these inequalities with O3, we conclude $x^{n+1} < y^{n+1}$. So the statement is true for $n + 1$.

Therefore, by the Principle of Mathematical Induction, we conclude if $0 < x < y$, then $x^n < y^n \forall n \in \mathbb{N}$.

5) (a) Suppose $0 < c < 1$.
   (i) Since $c > 0$ and $c < 1$, we have $c^2 < 1c = c$ by O5 and M3. Thus, $c^n < c$ for $n = 2$.

   (ii) Let $n \in \mathbb{N}$ with $n \geq 2$. Suppose that $c^n < c$ is true. Since $c > 0$, multiplying both sides by $c$ yields $c^{n+1} < c^2$ by O5, so since $c^2 < c$, we have $c^{n+1} < c$ by O3. So the statement is true for $n + 1$.

Therefore, by the Principle of Mathematical Induction, we conclude if $0 < c < 1$, then $c^n < c \forall n \in \mathbb{Z}$ with $n > 1$.

   (b) Suppose $c > 1$.
   (i) Since $c > 1$ and $1 > 0$, we have $c > 0$ by O3, so we get $c^2 > 1c = c$ by O5 and M3. Thus, $c^n > c$ for $n = 2$.

   (ii) Let $n \in \mathbb{N}$ with $n \geq 2$. Suppose that $c^n > c$ is true. Since $c > 0$, multiplying both sides by $c$ yields $c^{n+1} > c^2$ by O5, so since $c^2 > c$, we have $c^{n+1} > c$ by O3. So the statement is true for $n + 1$.

Therefore, by the Principle of Mathematical Induction, we conclude if $c > 1$, then $c^n > c \forall n \in \mathbb{Z}$ with $n > 1$.
6) Suppose $a^2 + b^2 = 0$. Using A4, we get $(a^2 + b^2) + (-b^2) = 0 + (-b^2)$. This implies $a^2 + (b^2 + (-b^2)) = -b^2$ by A1 and A3. Hence, we have $a^2 = a^2 + 0 = -b^2$ by A4 and A3, which tells us $a^2 = -b^2$. By using (iv) of Theorem 3.2 twice, we know $a^2 \geq 0$ and $b^2 \geq 0$ and then $-b^2 \leq 0$ by (i) of Theorem 3.2. Since $a^2 = -b^2 \leq 0$ and $a^2 \geq 0$, we must have $a^2 = 0$ by O2 and consequently, $-b^2 = 0$. Using (vi) of Theorem 3.1 twice on these equalities, we conclude $a = 0$ and $b = 0$. 