Homework 3 Solutions

3.3) (a) Notice by A4, we have $a+(-a)=0$ which implies $a=-(-a)$, then we can obtain $ab=-(-ab)$. Combining this with part (iii) of Theorem 3.1, the following occurs

$$(-a)(-b)=-(a(-b))=-((-b)a)=-(-ba)=-(-ab)=ab,$$

where we used the commutative law (M2) twice. Thus, we conclude $(-a)(-b)=ab$.

(b) Suppose $ac=bc$ and $c\neq 0$. Then by M4 $\exists c^{-1}\in\mathbb{R}$ so we can multiply to get $(ac)c^{-1}=(bc)c^{-1}$. So $a(cc^{-1})=b(cc^{-1})$ by M1, and this becomes $a1=b1$ by the inverse property M4. Thus, we conclude $a=b$ by M3.

3.4) (a) By (iv) in Theorem 3.2, we have $0\leq 1^2=1$, and since $0\neq 1$ by M3, we conclude $0<1$.

(b) Suppose $0<a<b$. Using (vi) of Theorem 3.2 twice, we have $a^{-1}>0$ and $b^{-1}>0$. Since $a<b$, we have $a^{-1}a<a^{-1}b$ by O5 and $1<a^{-1}b$ by M4. Then $1(b^{-1})<(a^{-1}b)b^{-1}$ by O5, so we have $b^{-1}<a^{-1}(bb^{-1})$ by M3 and M1. Finally, we can obtain $b^{-1}<a^{-1}\cdot 1=a^{-1}$ by M4 and M3. Therefore, we conclude $0<b^{-1}<a^{-1}$.

3.5) (b) Suppose $a,b\in\mathbb{R}$. We will consider the only two cases: (i) $|a|-|b|\geq 0$ and (ii) $|a|-|b|<0$.

Case (i): Suppose $|a|-|b|\geq 0$. Then, we must have $\Big||a|-|b|\Big|=|a|-|b|$ by definition of absolute value. By the Triangle Inequality, we get $|a|=|(a-b)+b|\leq|a-b|+|b|$. So, with subtraction by $|b|$, we obtain

$$\Big||a|-|b|\Big|=|a|-|b|\leq(|a-b|+|b|)-|b|=|a-b|.$$

Thus, $\Big||a|-|b|\Big|\leq|a-b|$.

Case (ii): Suppose $|a|-|b|<0$. Then, we must have $\Big||a|-|b|\Big|=-(|a|-|b|)=|b|-|a|$ by definition of absolute value. By the Triangle Inequality, we get $|b|=|(b-a)+a|\leq|b-a|+|a|$. So, with subtraction by $|a|$, we obtain

$$\Big||a|-|b|\Big|=|b|-|a|\leq(|b-a|+|a|)-|a|=|b-a|=|a-b|.$$

Thus, we conclude $\Big||a|-|b|\Big|\leq|a-b|$.

Since a and b were arbitrary, both cases give us $\Big||a|-|b|\Big|\leq|a-b|\ \forall a,b\in\mathbb{R}$.

3.6) (b) (i) This is true for $n=1$, since $|a_1|\leq|a_1|$ by equality.

(ii) Let $n \in \mathbb{N}$. Suppose that

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n| \tag{1}$$

is true. Then, by the Triangle Inequality, we obtain

$$|a_1 + a_2 + ... + a_n + a_{n+1}| = |(a_1 + a_2 + ... + a_n) + a_{n+1}| \leq |a_1 + a_2 + ... + a_n| + |a_{n+1}|$$

So, using the induction hypothesis (1), we conclude

$$|a_1 + a_2 + ... + a_n + a_{n+1}| \leq |a_1| + |a_2| + ... + |a_n| + |a_{n+1}|,$$

and the statement is true for $n + 1$.

Therefore, by the Principle of Mathematical Induction, we conclude

$$|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n| \quad \forall n \in \mathbb{N}$$

Worksheet 2 Solutions

1) Suppose $a \leq b$ and $c \leq d$. Since $a \leq b$, we have $a+c \leq b+c$ by O4. Similarly, since $c \leq d$, we obtain $b+c \leq b+d$ with O4 and A2. Using both of these inequalities along with O3, we conclude $a+c \leq b+d$.

2) Suppose $0 \leq a \leq b$ and $0 \leq c \leq d$. Since $a \leq b$ and $0 \leq c$, we get $ac \leq bc$ by O5. Similarly, since $c \leq d$ and $0 \leq b$, we obtain $bc \leq bd$ with O5 and M2. Combining these inequalities with O3, we conclude $ac \leq bd$.

3) Suppose $x > 0$, $y > 0$, and $x^2 < y^2$, but $x \geq y$. Since $x > 0$ and $x \geq y$, we have $x^2 \geq xy$ by O5 and M2. Similarly, since $y > 0$ and $x \leq y$, we get $xy \geq y^2$ with O5. Combining these inequalities with O3, we obtain $x^2 \geq y^2$, but we assumed $x^2 < y^2$. Contradiction!

Therefore, if $x > 0$, $y > 0$, and $x^2 < y^2$, then $x < y$.

4) Suppose $0 < x < y$.

(i) This is true for $n = 1$, since $x < y$ by assumption.

(ii) Let $n \in \mathbb{N}$. Suppose that $x^n > y^n$ is true. Since $x > 0$, multiplying both sides by x yields $x^{n+1} < y^n$ by O5 and M2. Since $y^n > 0$, through repeated use of (iii) of Theorem 3.2, and $x < y$, we have $xy^n < y^{n+1}$ by O5. Combining these inequalities with O3, we conclude $x^{n+1} < y^{n+1}$. So the statement is true for $n+1$.

Therefore, by the Principle of Mathematical Induction, we conclude if $0 < x < y$, then $x^n < y^n$ $\forall n \in \mathbb{N}$.

5) (a) Suppose $0 < c < 1$.

(i) Since $c > 0$ and $c < 1$, we have $c^2 < 1c = c$ by O5 and M3. Thus, $c^n < c$ for $n = 2$.

(ii) Let $n \in \mathbb{N}$ with $n \geq 2$. Suppose that $c^n < c$ is true. Since $c > 0$, multiplying both sides by c yields $c^{n+1} < c^2$ by O5, so since $c^2 < c$, we have $c^{n+1} < c$ by O3. So the statement is true for $n+1$.

Therefore, by the Principle of Mathematical Induction, we conclude if $0 < c < 1$, then $c^n < c$ $\forall n \in \mathbb{Z}$ with $n > 1$.

(b) Suppose $c > 1$.

(i) Since $c > 1$ and $1 > 0$, we have $c > 0$ by O3, so we get $c^2 > 1c = c$ by O5 and M3. Thus, $c^n > c$ for $n = 2$.

(ii) Let $n \in \mathbb{N}$ with $n \geq 2$. Suppose that $c^n > c$ is true. Since $c > 0$, multiplying both sides by c yields $c^{n+1} > c^2$ by O5, so since $c^2 > c$, we have $c^{n+1} > c$ by O3. So the statement is true for $n+1$.

Therefore, by the Principle of Mathematical Induction, we conclude if $c > 1$, then $c^n > c$ $\forall n \in \mathbb{Z}$ with $n > 1$.

6) Suppose $a^2+b^2=0$. Using A4, we get $(a^2+b^2)+(-b^2)=0+(-b^2)$. This implies $a^2+(b^2+(-b^2))=-b^2$ by A1 and A3. Hence, we have $a^2=a^2+0=-b^2$ by A4 and A3, which tells us $a^2=-b^2$. By using (iv) of Theorem 3.2 twice, we know $a^2\geq 0$ and $b^2\geq 0$ and then $-b^2\leq 0$ by (i) of Theorem 3.2. Since $a^2=-b^2\leq 0$ and $a^2\geq 0$, we must have $a^2=0$ by O2 and consequently, $-b^2=0$. Using (vi) of Theorem 3.1 twice on these equalities, we conclude $a=0$ and $b=0$.