Homework 4 Solutions

4.1e-4.4e) Consider

$$S := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, ... \right\}.$$

1) Upper bounds: 1, 2, 19, ...
2) Lower bounds: 0, -1, -$\frac{5}{3}$, ...
3) sup $S = 1$, max $S = 1$
4) inf $S = 0$, min S does not exist

4.1g-4.4g) Consider

$$S := [0,1] \cup [2,3] = \{x : 0 \leq x \leq 1 \text{ or } 2 \leq x \leq 3\}.$$

1) Upper bounds: 3, π, 100, ...
2) Lower bounds: 0, -7, -51, ...
3) sup $S = 3$, max $S = 3$
4) inf $S = 0$, min $S = 0$

4.3j-4.4j) Consider

$$S := \left\{ 1 - \frac{1}{3^n} : n \in \mathbb{N} \right\} = \left\{ 1 - \frac{1}{3}, 1 - \frac{1}{9}, 1 - \frac{1}{27}, ... \right\}.$$

3) sup $S = 1$, max S does not exist
4) inf $S = \frac{2}{3}$, min $S = \frac{2}{3}$

4.3s-4.4s) Consider

$$S := \left\{ \frac{1}{3^n} : n \text{ is prime} \right\} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, ... \right\}.$$

3) sup $S = \frac{1}{2}$, max $S = \frac{1}{2}$
4) inf S =0, min S does not exist

Notes:

(1) In 3j, we could prove that sup $S = 1$ using the Archimedean Property and the fact that $3^n > n \ \ \forall n \in \mathbb{N}$.

(2) In 4s, we could prove that inf $S = 0$ using the Archimedean Property and the fact that there are infinitely many primes.

4.7) (b) Suppose $S, T \subset \mathbb{R}$ are nonempty bounded sets. Without loss of generality, we assume sup $S \geq$ sup T, so $\max\{\sup S, \sup T\} =$ sup S. Since $S \subset S \cup T$, we have sup $S \leq$ sup $(S \cup T)$ by problem 4.7a.

If $x \in S \cup T$, then either $x \in S$ or $x \in T$. If $x \in S$, then $x \leq \sup S$. If $x \in T$, then $x \leq \sup T \leq \sup S$. Thus, since $x \in S \cup T$ was arbitrary, we have $x \leq \sup S \ \forall x \in S \cup T$. Hence, $\sup S$ is an upper bound, and we obtain $\sup (S \cup T) \leq \sup S$.

Therefore, since we have $\sup S \leq \sup (S \cup T)$ and $\sup (S \cup T) \leq \sup S$, we conclude $\sup (S \cup T) = \sup S = \max\{\sup S, \sup T\}$.

4.8) (b) Suppose $S, T \subset \mathbb{R}$ are nonempty with $s \leq t \ \ \forall s \in \mathbb{S}$ and $\forall t \in T$. Let $t \in T$. Since $s \leq t \ \ \forall s \in S$, t is an upper bound for S, so $\sup S \leq t$. Since $t \in T$ was arbitrary, we have $\sup S \leq t \ \ \forall t \in T$. Therefore, $\sup S$ is a lower bound for T, and we conclude $\sup S \leq \inf T$.

Worksheet 2 Solutions

7) Suppose $a, b \in \mathbb{R}$ with $a \geq 0$ and $b \geq 0$. Since $(a-b)^2 \geq 0$, we have $a^2 - 2ab + b^2 \geq 0$. Adding $4ab$ to both sides, gives us

$$a^2 + 2ab + b^2 \geq 4ab \Rightarrow (a+b)^2 \geq 4ab.$$

Taking the positive square root of both sides, we have

$$a + b \geq 2\sqrt{ab} \Rightarrow \sqrt{ab} \leq \frac{a+b}{2}.$$

Thus, we conclude

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

2) Suppose $r \in \mathbb{Q}$. Then we have two cases: (1) $r \leq 0$ or (2) $r > 0$.

For case (1), suppose $r \leq 0$. Then, we can easily choose $n = 1 \in \mathbb{N}$ since $r \leq 0 < 1$, and we obtain $r < n$.

For case (2), suppose $r > 0$. Then, by the Archimedean Property (with $a = 1 > 0$ and $b = r > 0$), $\exists n \in \mathbb{N}$ such that $r < 1n = n$. Thus, we have an $n \in \mathbb{N}$ with $r < n$.

Therefore, we conclude if $r \in \mathbb{Q}$, then $\exists n \in \mathbb{N}$ with $r < n$.

Note: This shows (with a little work) that $\mathbb{Q}$ satisfies the Archimedean Property.