Homework 5 Solutions

4.10) Suppose $a > 0$. By two applications of the Archimedean Property, $\exists m_1, m_2 \in \mathbb{N}$ such that $a < m_1$ and $\frac{1}{a} < m_2$. Choose $n = \max\{m_1, m_2\}$, so $n \geq m_1$ and $n \geq m_2$. Then, we must have $a < n$ and $\frac{1}{a} < n$. Rearranging the second inequality and combining, we obtain $\frac{1}{n} < a < n$.

Therefore, if $a > 0$, then $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

4.11) Suppose $a, b \in \mathbb{R}$ where $a < b$, but there are only a finite number of rationals in the interval (a, b). Denote these finite number of rationals by the set $R = \{r_1, r_2, ..., r_n\}$ for some $n \in \mathbb{N}$. Since R is finite, the maximum exists, so let $m = \max R$ where $a < m < b$ by definition. By the Denseness of $\mathbb{Q}$, $\exists r \in \mathbb{Q}$ such that $m < r < b$, which tells us $a < m < r < b$. But m is the maximum rational number of all the rationals in (a, b). Contradiction!

Therefore, if $a, b \in \mathbb{R}$ with $a < b$, there are infinitely many rational between a and b.

4.12) Suppose that $a, b \in \mathbb{R}$ with $a < b$. Subtracting $\sqrt{2}$ to both sides of the inequality, we get $a - \sqrt{2} < b - \sqrt{2}$. By the Denseness of $\mathbb{Q}$, $\exists r \in \mathbb{Q}$ with $a - \sqrt{2} < r < b - \sqrt{2}$. Adding $\sqrt{2}$ to all sides, we obtain $a < r + \sqrt{2} < b$. Let $x = r + \sqrt{2}$, and note that $x \in \mathbb{I}$ since $r \in \mathbb{Q}$ and $\sqrt{2} \in \mathbb{I}$. Thus, we found $x \in \mathbb{I}$ with $a < x < b$.

Therefore, if $a < b$, then $\exists x \in \mathbb{I}$ such that $a < x < b$.

Note: We just showed that the irrationals are also dense in $\mathbb{R}$. Moreover, one can show that there are infinitely many irrationals between any two numbers, analogous to the statement in problem 4.11.

4.14) (a) Let $A, B \subset \mathbb{R}$ are nonempty and $S := \{a + b : a \in A, b \in B\}$.

Let $s \in S$. So, $\exists a \in A$ and $\exists b \in B$ such that $s = a + b$. Then, $s = a + b \leq \sup A + \sup B$, and $\sup A + \sup B$ is an upper bound for S. Thus, we have

$$\sup S \leq \sup A + \sup B. \tag{1}$$

Let $a \in A$ and $b \in B$. Then, $a + b \leq \sup S$, so $a \leq \sup S - b$. Since a was arbitrary, we have $a \leq \sup S - b \ \ \forall a \in A$ and $\sup S - b$ is an upper bound for A, so $\sup A \leq \sup S - b$. Since b was arbitrary, we obtain

$$\sup A \leq \sup S - b \ \ \forall b \in B.$$

Rewriting this, we have $b \leq \sup S - \sup A \ \ \forall b \in B$. and $\sup S - \sup A$ is an upper bound for B, so $\sup B \leq \sup S - \sup A$. Hence, rearranging this equation gives us

$$\sup A + \sup B \leq \sup S. \tag{2}$$

Therefore, by (1) and (2), we conclude that

$$\sup S = \sup A + \sup B.$$

4.15) Let $a, b \in \mathbb{R}$. Suppose that $a \leq b + \frac{1}{n} \quad \forall n \in \mathbb{N}$, but $a > b$. Then $a - b > 0$, and $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < a - b$ by the Archimedean Property. Thus, rewriting yields $b + \frac{1}{n} < a$ for some $n \in \mathbb{N}$, but we assumed $a \leq b + \frac{1}{n} \quad \forall n \in \mathbb{N}$. Contradiction!

Therefore, we conclude if $a \leq b + \frac{1}{n} \quad \forall n \in \mathbb{N}$, then $a \leq b$.

Worksheet 3 Solutions

1) Define $E := \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 < 2\}$. Let $r \in E$. Since $r^2 < 2 < 4$ and r was arbitrary, 4 must be an upper bound for E. Notice, that $1 \in E$ so $E \subset \mathbb{R}$ is nonempty. For the following steps (A-C), suppose that $s = \sup E$ exists.

A) If $s \in \mathbb{Q}$, then $s^2 \neq 2$ since we know that $\sqrt{2} \in \mathbb{I}$.

Now, define $r = \dfrac{2s+2}{s+2}$.

B) Suppose $s^2 > 2$. Then, through algebraic manipulation, we get

$$2s^2 > 4 \Rightarrow 4s^2 + 8s + 4 > 2s^2 + 8s + 8 \Rightarrow (2s+2)^2 > 2(s^2 + 4s + 4) \Rightarrow (2s+2)^2 > 2(s+2)^2$$

$$\Rightarrow \frac{(2s+2)^2}{(s+2)^2} > 2 \Rightarrow r^2 = \left(\frac{2s+2}{s+2}\right)^2 > 2.$$

Thus, we have $r^2 > 2$

Clearly, we have $r > 0$ since $s > 0$. Also, we obtain

$$2 < s^2 \Rightarrow 2s + 2 < s^2 + 2s = s(s+2) \Rightarrow \frac{2s+2}{s+2} < s \Rightarrow r < s.$$

Hence, $0 < r < s$.

Therefore, we conclude if $s^2 > 2$, then $r^2 > 2$ and $0 < r < s$. In this scenario, r is an upper bound of E with $r < s$, so s cannot be the least upper bound (i.e. $s \neq \sup E$).

C) Suppose $s^2 < 2$. Then, through algebraic manipulation, we get

$$2s^2 < 4 \Rightarrow 4s^2 + 8s + 4 < 2s^2 + 8s + 8 \Rightarrow (2s+2)^2 < 2(s^2 + 4s + 4) \Rightarrow (2s+2)^2 < 2(s+2)^2$$

$$\Rightarrow \frac{(2s+2)^2}{(s+2)^2} < 2 \Rightarrow r^2 = \left(\frac{2s+2}{s+2}\right)^2 < 2.$$

Thus, we have $r^2 < 2$.

Since $s^2 < 2$, we can also obtain

$$2 > s^2 \Rightarrow 2s + 2 > s^2 + 2s = s(s+2) \Rightarrow \frac{2s+2}{s+2} > s \Rightarrow r > s.$$

Hence, $s < r$.

Therefore, we conclude if $s^2 < 2$, then $r^2 < 2$ and $s < r$. But, clearly $r > 0$ as well, so $r \in E$ with $s < r$, and in this scenario, s cannot be an upper bound much less the least upper bound (i.e. $s \neq \sup E$).

Note: If $s = \sup E$ exists in $\mathbb{Q}$, then one of the three cases must occur: A) $s^2 = 2$, B) $s^2 > 2$, or C) $s^2 < 2$. We just went through all these three cases above and showed how each led to a contradiction. This proves that $s = \sup E$ is not a rational number.

But, by the Completeness Axiom, $s = \sup E \in \mathbb{R}$ exists, so we must have $s \in \mathbb{I}$ since $s \notin \mathbb{Q}$. Also, this irrational number will still not satisfy B) or C), but it will satisfy A) (i.e. $s^2 = 2$). The next problem goes over this fact.

2) Let $S := \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 3\}$.

A) Clearly, $1 \in S$, so S is nonempty. Also, 2 is a upper bound for S since $\forall x \in S$, $x^2 < 3 < 4$ implies $x < 2$.

By the Completeness Axiom, $\sup S \in \mathbb{R}$ exists. Now, let $t = \sup S$.

B) Suppose $t^2 < 3$. So, we have

$$t^2 < 3 \Rightarrow t < \sqrt{3} \Rightarrow \sqrt{3} - t > 0.$$

Then, by the Archimedean Property, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \sqrt{3} - t$. Rewriting this inequality gives us

$$\frac{1}{n} < \sqrt{3} - t \Rightarrow t + \frac{1}{n} < \sqrt{3} \Rightarrow \left(t + \frac{1}{n}\right)^2 < 3.$$

Thus, we conclude if $t^2 < 3$, $\exists n \in \mathbb{N}$ such that $(t + \frac{1}{n})^2 < 3$. In this scenario, $t + \frac{1}{n} \in S$ and $t < t + \frac{1}{n}$, so t cannot be an upper bound for S much less the least upper bound (i.e. $t \neq \sup S$).

C) Suppose $t^2 > 3$. So, we have

$$t^2 > 3 \Rightarrow t > \sqrt{3} \Rightarrow t - \sqrt{3} > 0.$$

Then, by the Archimedean Property, $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < t - \sqrt{3}$. Rewriting this inequality gives us

$$\frac{1}{m} < t - \sqrt{3} \Rightarrow t - \frac{1}{m} > \sqrt{3} \Rightarrow \left(t - \frac{1}{m}\right)^2 > 3.$$

Thus, we conclude if $t^2 > 3$, $\exists m \in \mathbb{N}$ such that $(t - \frac{1}{m})^2 > 3$. In this scenario, $t - \frac{1}{m}$ is an upper bound of S and $t - \frac{1}{n} < t$, so t cannot be the least upper bound (i.e. $t \neq \sup S$).

Note: If $t = \sup S$ exists in $\mathbb{R}$, then one of the three cases must occur: A) $t^2 = 3$, B) $t^2 > 3$, or C) $t^2 < 2$. We just showed above that cases B) and C) lead to contradictions. Thus, by the process of elimination, we must have $t^2 = 3$. So we just proved that $\exists t \in \mathbb{R}$ such that $t^2 = 3$. Moreover, in light of problem 1), t must be irrational, which can be proven by other means as well.