Homework 7 Solutions

8.2) (a) Let $\epsilon > 0$ be given. Notice that

$$|s_n - s| = \left|\frac{n}{n^2+1} - 0\right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n}$$

So, we have

$$|s_n - s| < \epsilon \Leftrightarrow \frac{n}{n^2+1} < \frac{1}{n} < \epsilon \Leftrightarrow \frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$$

Choose $N = \frac{1}{\epsilon}$. Thus, $\forall n > N$, then $\left|\frac{n}{n^2+1} - 0\right| < \epsilon$.

Therefore, we conclude $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$.

(c) Let $\epsilon > 0$ be given. Notice that

$$|s_n - s| = \left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| = \left|\frac{7(4n+3) - 4(7n-5)}{7(7n-5)}\right| = \frac{41}{7(7n-5)}$$

So, we have

$$|s_n - s| < \epsilon \Leftrightarrow \frac{41}{7(7n-5)} < \epsilon \Leftrightarrow \frac{41}{7\epsilon} < 7n - 5 \Leftrightarrow n > \frac{1}{7}\left(\frac{41}{7\epsilon} + 5\right)$$

Choose $N = \frac{1}{7}\left(\frac{41}{7\epsilon} + 5\right)$. Thus, $\forall n > N$, then $\left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| < \epsilon$.

Therefore, we conclude $\lim_{n\to\infty} \frac{4n+3}{7n-5} = \frac{4}{7}$.

(d) Let $\epsilon > 0$ be given. Notice that

$$|s_n - s| = \left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| = \left|\frac{5(2n+4) - 2(5n+2)}{5(5n+2)}\right| = \frac{16}{5(5n+2)}$$

So, we have

$$|s_n - s| < \epsilon \Leftrightarrow \frac{16}{5(5n+2)} < \epsilon \Leftrightarrow \frac{16}{5\epsilon} < 5n - 2 \Leftrightarrow n > \frac{1}{5}\left(\frac{16}{5\epsilon} + 2\right)$$

Choose $N = \frac{1}{5}\left(\frac{16}{5\epsilon} + 2\right)$. Thus, $\forall n > N$, then $\left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| < \epsilon$.

Therefore, we conclude $\lim_{n\to\infty} \frac{2n+4}{5n+2} = \frac{2}{5}$.

(e) Let $\epsilon > 0$ be given. Notice that

$$|s_n - s| = \left|\frac{\sin n}{n} - 0\right| = \frac{|\sin n|}{n} \le \frac{1}{n}$$

So, we have

$$|s_n - s| < \epsilon \Leftrightarrow \frac{|\sin n|}{n} \le \frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$$

Choose $N = \frac{1}{\epsilon}$. Thus, $\forall n > N$, then $\left|\frac{\sin n}{n} - 0\right| < \epsilon$.

Therefore, we conclude $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

8.3) (a) Suppose that we have $\{s_n\}$ a sequence of positive numbers with $\lim_{n\to\infty} s_n = 0$. Let $\epsilon > 0$ be given. Notice that

$$|\sqrt{s_n} - 0)| = \sqrt{s_n} < \epsilon \Leftrightarrow s_n < \epsilon^2 \Leftrightarrow |s_n - 0| < \epsilon^2$$

Since $\lim_{n\to\infty} s_n = 0$, $\exists N_1 \in \mathbb{R}$ such that $\forall n > N_1$, then $|s_n - 0| < \epsilon^2$. Choose $N = N_1$. Then, $\forall n > N$, then $|\sqrt{s_n} - 0| < \epsilon$.

Therefore, we conclude if $\{s_n\}$ is a sequence of positive numbers with $\lim_{n\to\infty} s_n = 0$, then $\lim_{n\to\infty} \sqrt{s_n} = 0$.

8.4) Suppose $\{t_n\}$ is a bounded sequence and $\{s_n\}$ is a sequence with $\lim_{n\to\infty} s_n = 0$. Let $\epsilon > 0$ be given. Notice

$$|s_n t_n - 0| = |s_n t_n| = |s_n||t_n|.$$

Since $\{t_n\}$ is bounded, $\exists M > 0$ such that $|t_n| \leq M$ $\forall n \in \mathbb{N}$. Since $\lim_{n\to\infty} s_n = 0$, $\exists N_1 \in \mathbb{R}$ such that $\forall n > N_1$, then $|s_n - 0| < \frac{\epsilon}{M}$. Choose $N = N_1$. Thus, $\forall n > N$, then

$$|s_n t_n - 0| = |s_n||t_n| \leq \frac{\epsilon}{M} M = \epsilon.$$

Therefore, we conclude if $\{t_n\}$ is a bounded sequence and $\{s_n\}$ is a sequence with $\lim_{n\to\infty} s_n = 0$, then $\lim_{n\to\infty} s_n t_n = 0$.

8.5) (b) Suppose $\{s_n\}$ and $\{t_n\}$ are sequences such that $|s_n| \leq t_n$ $\forall n \in \mathbb{N}$ with $\lim_{n\to\infty} t_n = 0$. Since $|s_n| \leq t_n$ $\forall n \in \mathbb{N}$, we have

$$-t_n \leq s_n \leq t_n \quad \forall n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} t_n = 0$ and consequently, $\lim_{n\to\infty} -t_n = 0$, we conclude $\lim_{n\to\infty} s_n = 0$ by the Squeeze Theorem.

8.6) (a) (Remark: since this is a biconditional we have to prove both implications $\Leftarrow$ and $\Rightarrow$)

($\Rightarrow$) Suppose $\lim_{n\to\infty} s_n = 0$. Let $\epsilon > 0$ be given. Then, $\exists N \in \mathbb{R}$ such that $\forall n > N$, then $|s_n - 0| < \epsilon$. We choose this N given to us. Thus, $\forall n > N$, then

$$||s_n| - 0| = |s_n| < \epsilon.$$

Therefore, we conclude if $\lim_{n\to\infty} s_n = 0$, then $\lim_{n\to\infty} |s_n| = 0$.

($\Leftarrow$) Suppose $\lim_{n\to\infty} |s_n| = 0$. Since $-|s_n| \leq s_n \leq |s_n|$ $\forall n \in \mathbb{N}$, $\lim_{n\to\infty} |s_n| = 0$, and consequently, $\lim_{n\to\infty} -|s_n| = 0$, we conclude $\lim_{n\to\infty} s_n = 0$ by the Squeeze Theorem.

8.7) (a) Consider the sequence $\{s_n\}$ with $s_n = \cos\left(\frac{n\pi}{3}\right)$ $\forall n \in \mathbb{N}$. Notice that

$$s_n = \begin{cases} 1 & \text{if } 6|n \\ \frac{1}{2} & \text{if } 6|(n-1) \text{ or } 6|(n-5) \\ -\frac{1}{2} & \text{if } 6|(n-2) \text{ or } 6|(n-4) \\ -1 & \text{if } 6|(n-3) \end{cases}$$

Choose $\epsilon = 1$, and let $N \in \mathbb{R}$ be given. By Archimedian Property, $\exists n^* \in \mathbb{N}$ such that $n* > N$. Now choose $n > n* > N$ such that it's the next natural number which satisfies $6|(n-1)$ (which is always possible since there are an infinite number of positive solutions to $n - 1 = 6k$ where k can be any integer). Then, we have

$$|s_n - 1| = |-1-1| = 2 \geq 1 = \epsilon.$$

So $|s_n - 1| \geq \epsilon$.

Thus, we conclude $\lim_{n\to\infty} s_n \neq 1$

(c) This will be an optional homework problem.