Homework 8 Solutions

Theorem (Squeeze Theorem)**.** *If $\{a_n\}$, $\{b_n\}$, and $\{s_n\}$ are sequences such that $a_n \le s_n \le b_n$ $\forall n > N$ where $N \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$. Then, we have $\lim_{n\to\infty} s_n = L$.*

9.3) Supose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Consider the following sequence

$$s_n = \frac{a_n^3 + 4a_n}{b_n^2 + 1}.$$

Then, we have

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a_n^3 + 4a_n}{b_n^2 + 1} = \frac{\lim_{n\to\infty} a_n^3 + 4a_n}{\lim_{n\to\infty} b_n^2 + 1} = \frac{\lim_{n\to\infty} a_n^3 + \lim_{n\to\infty} 4a_n}{\lim_{n\to\infty} b_n^2 + \lim_{n\to\infty} 1},$$

using Theorems 9.6 and 9.3. Continuing this process by using Theorems 9.4 and 9.2, we obtain

$$\lim_{n\to\infty} s_n = \frac{\left(\lim_{n\to\infty} a_n\right)^3 + 4\lim_{n\to\infty} a_n}{\left(\lim_{n\to\infty} b_n\right)^2 + 1} = \frac{a^3 + 4a}{b^2 + 1}.$$

Thus, we conclude

$$\lim_{n\to\infty} s_n = \frac{a^3 + 4a}{b^2 + 1}.$$

9.4) Let $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ (i.e. a recursion relation).

(a) $s_1 = 1$, $s_2 = \sqrt{2}$, $s_3 = \sqrt{\sqrt{2} + 1}$, and $s_4 = \sqrt{\sqrt{\sqrt{2} + 1} + 1}$.

(b) Suppose that $\{s_n\}$ converges, and let $\lim_{n\to\infty} s_n = s$. Taking the limit of both sides of the recursion relation, we obtain

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \sqrt{s_n + 1} \Rightarrow s = \sqrt{s+1} \Rightarrow s^2 = s + 1 \Rightarrow s^2 - s - 1 = 0 \Rightarrow s = \frac{1 \pm \sqrt{5}}{2}.$$

Since $s_n > 0$ $\forall n \in \mathbb{N}$, we conclude

$$s = \frac{1 + \sqrt{5}}{2}.$$

9.5) Let $t_1 = 1$ and $t_{n+1} = \frac{t_n^2 + 2}{2t_n}$ (i.e. a recursion relation). Suppose that $\{t_n\}$ converges, and let $\lim_{n\to\infty} t_n = t$. Taking the limit of both sides of the recursion relation, we obtain

$$\lim_{n\to\infty} t_{n+1} = \lim_{n\to\infty} \frac{t_n^2 + 2}{2t_n} \Rightarrow t = \frac{t^2 + 2}{2t} \Rightarrow 2t^2 = t^2 + 2 \Rightarrow t = \pm\sqrt{2}.$$

Since $t_n > 0$ $\forall n \in \mathbb{N}$, we conclude $t = \sqrt{2}$.

Worksheet 4 Solutions

1) Supose that $\lim_{n\to\infty} s_n = s$. Let $\epsilon > 0$ be given. Since $\{s_n\}$ converges to s, $\exists N \in \mathbb{R}$ such that $\forall n > N$, then $|s_n - s| < \epsilon$. We choose this N. Thus, we have $\forall n > N$, then $\Big||s_n| - |s|\Big| \leq |s_n - s| < \epsilon$, (The inequality used here, which is on your handout, is sometimes referred to as the Reverse Triangle Inequality).

Therefore, we conclude $\lim_{n\to\infty} |s_n| = |s|$.

2) Suppose that $\lim_{n\to\infty} s_n = L$ and f is continuous at L. Let $\epsilon > 0$ be given. Since f is continuous at L, by definition, $\exists \delta > 0$ such that

$$\text{if } |x - L| < \delta \Rightarrow |f(x) - f(L)| < \epsilon. \tag{1}$$

Also, since $\lim_{n\to\infty} s_n = L$, $\exists N \in \mathbb{R}$ such that

$$\forall n > N \Rightarrow |s_n - L| < \delta. \tag{2}$$

Thus, by combining (1) and (2), we have

$$\forall n > N \Rightarrow |f(s_n) - f(L)| < \epsilon.$$

Therefore, we conclude $\lim_{n\to\infty} f(s_n) = f(L)$.

3) Suppose that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$

(a) Then we have $\lim_{n\to\infty} (s_n - s) = 0$ and $\lim_{n\to\infty} (t_n - t) = 0$ by Theorems 9.3.

For the next part, one can show the following is true using the definition of the limit (which is an optional homework assignment):

Proposition 1. *If* $\lim_{n\to\infty} s_n = 0$ *and* $\lim_{n\to\infty} t_n = 0$, *then* $\lim_{n\to\infty} s_n t_n = 0$.

(b) By Proposition 1 and part (a), we have $\lim_{n\to\infty} (s_n - s)(t_n - t) = 0$. Thus, we can obtain through algebra $\lim_{n\to\infty} (s_n t_n - s t_n - s t_n + st) = 0$

(c) Using addition by zero three times, we get

$$\lim_{n\to\infty} s_n t_n = \lim_{n\to\infty} s_n t_n - s t_n - s_n t + st + s t_n + s_n t - st$$
$$= \lim_{n\to\infty} (s_n t_n - s t_n - s t_n + st) + \lim_{n\to\infty} s t_n + \lim_{n\to\infty} s_n t - \lim_{n\to\infty} st,$$

where the last equality uses Theorem 9.3 many times.

(d) Finally, starting with part (c), we arrive at

$$\lim_{n\to\infty} s_n t_n = 0 + s \lim_{n\to\infty} t_n + t \lim_{n\to\infty} s_n - st = st + st - st = st,$$

using Theorem 9.2 twice.

Remark: This gives an alternate proof of Theorem 9.4.

4) Suppose that $\{s_n\}$ is bounded. Since $\{s_n\}$ is bounded, we have $\exists M > 0$ such that $|s_n| \leq M \ \forall n \in \mathbb{N}$. So we have

$$-M \leq s_n \leq M \quad \forall n \in \mathbb{N} \Leftrightarrow \frac{-M}{n} \leq \frac{s_n}{n} \leq \frac{M}{n} \quad \forall n \in \mathbb{N}.$$

Since $\displaystyle\lim_{n\to\infty} \frac{-M}{n} = \lim_{n\to\infty} \frac{M}{n} = 0$, we have $\displaystyle\lim_{n\to\infty} \frac{s_n}{n} = 0$, by the Squeeze Theorem.

Therefore, if $\{s_n\}$ is bounded, then $\displaystyle\lim_{n\to\infty} \frac{s_n}{n} = 0$.