Homework 9 Solutions

Theorem (Squeeze Theorem)**.** *If $\{a_n\}$, $\{b_n\}$, and $\{s_n\}$ are sequences such that $a_n \leq s_n \leq b_n$ $\forall n > N$ where $N \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$. Then, we have $\lim_{n\to\infty} s_n = L$.*

9.9) Suppose that there exists N_0 such that $s_n \leq t_n \;\; \forall n > N_0$.

(a) Also, suppose $\lim_{n\to\infty} s_n = \infty$. Let $M > 0$ be given. Since $\lim_{n\to\infty} s_n = \infty$, $\exists N_1 \in \mathbb{R}$ such that $\forall n \geq N_1$, then $s_n > M$, by definition. Choose $N = \max\{N_1, N_0\}$. Thus, $\forall n > N$, then $t_n \geq s_n > M$.

Therefore, we conclude $\lim_{n\to\infty} t_n = \infty$.

9.11) (c) Suppose $\lim_{n\to\infty} s_n = \infty$ and t_n is a bounded sequence. Let $M > 0$ be given. Since $\{t_n\}$ is bounded, it must have a lower bound l, so $t_n \geq l \;\; \forall n \in \mathbb{N}$. Since $\lim_{n\to\infty} s_n = \infty$, $\exists N \in \mathbb{R}$ such that $\forall n \geq N$, then $s_n > M - l$, by definition. We choose this N. Thus, $\forall n > N$, then $s_n + t_n > M - l + l = M$.

Therefore, we conclude $\lim_{n\to\infty} s_n + t_n = \infty$.

9.12) Suppose $s_n \neq 0$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$.

(a) Suppose $L < 1$. By the Denseness of $\mathbb{Q}$, $\exists a \in \mathbb{Q}$ such that $L < a < 1$. Since $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$ and $a - L > 0$, $\exists N^* \in \mathbb{R}$ such that

$$\left(\forall n > N^* \Rightarrow \left|\frac{s_{n+1}}{s_n} - L\right| < a - L\right) \Leftrightarrow \frac{|s_{n+1}|}{|s_n|} < a \;\; \forall n > N^*.$$

Define $N = \max\{\lceil N^* \rceil, 1\}$ so $N > N^*$ and $N \in \mathbb{N}$. Rewriting above inequality gives us

$$|s_{n+1}| < a|s_n| \;\; \forall n > N. \tag{1}$$

Fix $n \in \mathbb{N}$ with $n > N$, then repeated use of (1) we get

$$|s_n| < a|s_{n-1}| < a^2|s_{n-2}| < ... < a^{n-N}|s_N|$$

Since $n > N$ was arbitrary, we have

$$0 < |s_n| < a^{n-N}|s_N| \;\; \forall n > N.$$

Since $\lim_{n\to\infty} a^{n-N}|s_N| = 0$ by Theorem 9.7 and clearly $\lim_{n\to\infty} 0 = 0$, we have $\lim_{n\to\infty} |s_n| = 0$. By HW 8.6 (a), we conclude $\lim_{n\to\infty} s_n = 0$

(b) Suppose $L > 1$. Consider a new sequence $\{t_n\} = \frac{1}{|s_n|}$, so we have

$$\lim_{n\to\infty} \left|\frac{t_{n+1}}{t_n}\right| = \lim_{n\to\infty} \left|\frac{s_n}{s_{n+1}}\right| = \frac{1}{L} < 1.$$

By HW 9.12 (a), we must have $\lim_{n\to\infty} t_n = 0$ where $t_n > 0\ \forall n \in \mathbb{N}$. Therefore, by Theorem 9.10, we conclude $\lim_{n\to\infty} |s_n| = \lim_{n\to\infty} \frac{1}{t_n} = \infty$.

9.17) Let $M > 0$ be given. Notice that

$$s_n > M \Leftrightarrow n^2 > M \Leftrightarrow n > \sqrt{M}$$

Choose $N = \sqrt{M}$. Thus, $\forall n > N$, then $n^2 > M$.

Therefore, we conclude $\lim_{n\to\infty} n^2 = \infty$.

Worksheet 4 Solutions

5) Notice we have

$$3^n < 3^n + 2^n < 2 \cdot 3^n \quad \forall n \in \mathbb{N}.$$

By taking the nth root of all sides and simplifying, we obtain

$$3 < (3^n + 2^n)^{\frac{1}{n}} < 2^{\frac{1}{n}} \cdot 3 \quad \forall n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} 3 = 3$ and $\lim_{n\to\infty} 2^{\frac{1}{n}} \cdot 3 = 3$ (Theorem 9.7), we conclude $\lim_{n\to\infty} (3^n + 2^n)^{\frac{1}{n}} = 3$ by the Squeeze Theorem.

6) Let $M > 0$ be given. Notice that

$$\frac{n^2+5}{n+3} > \frac{n^2}{n+3n} = \frac{n}{4}$$

So, we have

$$s_n > M \Leftrightarrow \frac{n^2+5}{n+3} > \frac{n}{4} > M \Leftrightarrow \frac{n}{4} > M \Leftrightarrow n > 4M$$

Choose $N = 4M$. Thus, $\forall n > N$, then $\frac{n^2+5}{n+3} > M$.

Therefore, we conclude $\lim_{n\to\infty} \frac{n^2+5}{n+3} = \infty$.