Now we apply the induction hypothesis $P_n$ to obtain

$$|\sin(n+1)x| \leq n|\sin x| + |\sin x| = (n+1)|\sin x|.$$

Thus $P_{n+1}$ holds. Finally, the result holds for all $n$ by mathematical induction. $\square$

## Exercises

1.1 Prove $1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all positive integers $n$.

1.2 Prove $3+11+\cdots+(8n-5) = 4n^2-n$ for all positive integers $n$.

1.3 Prove $1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2$ for all positive integers $n$.

1.4 **(a)** Guess a formula for $1+3+\cdots+(2n-1)$ by evaluating the sum for $n=1$, 2, 3, and 4. [For $n=1$, the sum is simply 1.]

**(b)** Prove your formula using mathematical induction.

1.5 Prove $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n} = 2-\frac{1}{2^n}$ for all positive integers $n$.

1.6 Prove $(11)^n - 4^n$ is divisible by 7 when $n$ is a positive integer.

1.7 Prove $7^n - 6n - 1$ is divisible by 36 for all positive integers $n$.

1.8 The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) $P_m$ is true, (ii) $P_{n+1}$ is true whenever $P_n$ is true and $n \geq m$.

**(a)** Prove $n^2 > n+1$ for all integers $n \geq 2$.

**(b)** Prove $n! > n^2$ for all integers $n \geq 4$. [Recall $n! = n(n-1)\cdots 2\cdot 1$; for example, $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.]

1.9 **(a)** Decide for which integers the inequality $2^n > n^2$ is true.

**(b)** Prove your claim in (a) by mathematical induction.

1.10 Prove $(2n+1)+(2n+3)+(2n+5)+\cdots+(4n-1) = 3n^2$ for all positive integers $n$.

1.11 For each $n \in \mathbb{N}$, let $P_n$ denote the assertion "$n^2+5n+1$ is an even integer."

**(a)** Prove $P_{n+1}$ is true whenever $P_n$ is true.

**(b)** For which $n$ is $P_n$ actually true? What is the moral of this exercise?

1.12 For $n \in \mathbb{N}$, let $n!$ [read "$n$ factorial"] denote the product $1 \cdot 2 \cdot 3 \cdots n$. Also let $0! = 1$ and define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for} \quad k = 0, 1, \ldots, n. \tag{1.1}$$

The *binomial theorem* asserts that

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= a^n + na^{n-1}b + \frac{1}{2}n(n-1)a^{n-2}b^2 + \cdots + nab^{n-1} + b^n.\end{aligned}$$

**(a)** Verify the binomial theorem for $n = 1$, 2, and 3.

**(b)** Show $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.

**(c)** Prove the binomial theorem using mathematical induction and part (b).

## §2 The Set $\mathbb{Q}$ of Rational Numbers

Small children first learn to add and to multiply positive integers. After subtraction is introduced, the need to expand the number system to include 0 and negative integers becomes apparent. At this point the world of numbers is enlarged to include the set $\mathbb{Z}$ of all *integers*. Thus we have $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$.

Soon the space $\mathbb{Z}$ also becomes inadequate when division is introduced. The solution is to enlarge the world of numbers to include all fractions. Accordingly, we study the space $\mathbb{Q}$ of all *rational numbers*, i.e., numbers of the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$. Note that $\mathbb{Q}$ contains all terminating decimals such as $1.492 = \frac{1{,}492}{1{,}000}$. The connection between decimals and real numbers is discussed in 10.3 on page 58 and in §16. The space $\mathbb{Q}$ is a highly satisfactory algebraic system in which the basic operations addition, multiplication, subtraction and division can be fully studied. No system is perfect, however, and $\mathbb{Q}$ is inadequate in some ways. In this section we will consider the defects of $\mathbb{Q}$. In the next section we will stress the good features of $\mathbb{Q}$ and then move on to the system of real numbers.

The set $\mathbb{Q}$ of rational numbers is a very nice algebraic system until one tries to solve equations like $x^2 = 2$. It turns out that no rational

**Proof**
In Example 1 we showed $b$ is a solution of $49x^4 - 56x^2 + 4 = 0$. By Theorem 2.2, the only possible rational solutions are

$$\pm 1, \pm 1/7, \pm 1/49, \pm 2, \pm 2/7, \pm 2/49, \pm 4, \pm 4/7, \pm 4/49.$$

To complete our proof, all we need to do is substitute these 18 candidates into the equation $49x^4 - 56x^2 + 4 = 0$. This prospect is so discouraging, however, that we choose to find a more clever approach. In Example 1, we also showed $12 = (4 - 7b^2)^2$. Now if $b$ were rational, then $4 - 7b^2$ would also be rational [Exercise 2.6], so the equation $12 = x^2$ would have a rational solution. But the only possible rational solutions to $x^2 - 12 = 0$ are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$, and these all can be eliminated by mentally substituting them into the equation. We conclude $4 - 7b^2$ cannot be rational, so $b$ cannot be rational. ■

As a practical matter, many or all of the rational candidates given by the Rational Zeros Theorem can be eliminated by approximating the quantity in question. It is nearly obvious that the values in Examples 2 through 5 are not integers, while all the rational candidates are. The number $b$ in Example 6 is approximately 0.2767; the nearest rational candidate is $+2/7$ which is approximately 0.2857.

It should be noted that not all irrational-looking expressions are actually irrational. See Exercise 2.7.

**2.4 Remark.**
While admiring the efficient Rational Zeros Theorem for finding rational zeros of polynomials with integer coefficients, you might wonder how one would find other zeros of these polynomials, or zeros of other functions. In §31, we will discuss the most well-known method, called Newton's method, and its cousin, the secant method. That discussion can be read now; only the proof of the theorem uses material from §31.

## Exercises

2.1 Show $\sqrt{3}$, $\sqrt{5}$, $\sqrt{7}$, $\sqrt{24}$, and $\sqrt{31}$ are not rational numbers.

2.2 Show $\sqrt[3]{2}$, $\sqrt[7]{5}$ and $\sqrt[4]{13}$ are not rational numbers.

2.3 Show $\sqrt{2+\sqrt{2}}$ is not a rational number.

2.4 Show $\sqrt[3]{5-\sqrt{3}}$ is not a rational number.

2.5 Show $[3+\sqrt{2}]^{2/3}$ is not a rational number.

2.6 In connection with Example 6, discuss why $4-7b^2$ is rational if $b$ is rational.

2.7 Show the following irrational-looking expressions are actually rational numbers: (a) $\sqrt{4+2\sqrt{3}}-\sqrt{3}$, and (b) $\sqrt{6+4\sqrt{2}}-\sqrt{2}$.

2.8 Find all rational solutions of the equation $x^8-4x^5+13x^3-7x+1=0$.

# §3 The Set $\mathbb{R}$ of Real Numbers

The set $\mathbb{Q}$ is probably the largest system of numbers with which you really feel comfortable. There are some subtleties but you have learned to cope with them. For example, $\mathbb{Q}$ is not simply the set of symbols $m/n$, where $m, n \in \mathbb{Z}, n \neq 0$, since we regard some pairs of different looking fractions as equal. For example, $\frac{2}{4}$ and $\frac{3}{6}$ represent the same element of $\mathbb{Q}$. A rigorous development of $\mathbb{Q}$ based on $\mathbb{Z}$, which in turn is based on $\mathbb{N}$, would require us to introduce the notion of equivalence classes. In this book we assume a familiarity with and understanding of $\mathbb{Q}$ as an algebraic system. However, in order to clarify exactly what we need to know about $\mathbb{Q}$, we set down some of its basic axioms and properties.

The basic algebraic operations in $\mathbb{Q}$ are addition and multiplication. Given a pair $a, b$ of rational numbers, the sum $a+b$ and the product $ab$ also represent rational numbers. Moreover, the following properties hold.

**A1.** $a+(b+c)=(a+b)+c$ for all $a, b, c$.
**A2.** $a+b=b+a$ for all $a, b$.
**A3.** $a+0=a$ for all $a$.
**A4.** For each $a$, there is an element $-a$ such that $a+(-a)=0$.
**M1.** $a(bc)=(ab)c$ for all $a, b, c$.
**M2.** $ab=ba$ for all $a, b$.

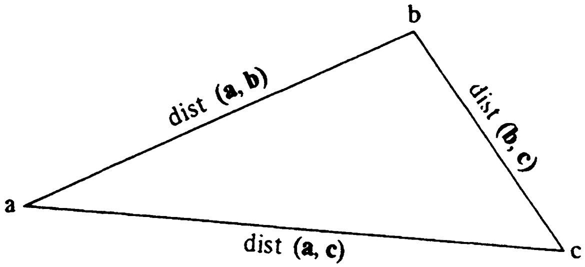


**FIGURE 3.2**

# Exercises

3.1 **(a)** Which of the properties A1–A4, M1–M4, DL, O1–O5 fail for $\mathbb{N}$?

**(b)** Which of these properties fail for $\mathbb{Z}$?

3.2 **(a)** The commutative law A2 was used in the proof of (ii) in Theorem 3.1. Where?

**(b)** The commutative law A2 was also used in the proof of (iii) in Theorem 3.1. Where?

3.3 Prove (iv) and (v) of Theorem 3.1.

3.4 Prove (v) and (vii) of Theorem 3.2.

3.5 **(a)** Show $|b| \leq a$ if and only if $-a \leq b \leq a$.

**(b)** Prove $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

3.6 **(a)** Prove $|a+b+c| \leq |a|+|b|+|c|$ for all $a, b, c \in \mathbb{R}$. *Hint*: Apply the triangle inequality twice. Do *not* consider eight cases.

**(b)** Use induction to prove

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

for $n$ numbers $a_1, a_2, \ldots, a_n$.

3.7 **(a)** Show $|b| < a$ if and only if $-a < b < a$.

**(b)** Show $|a - b| < c$ if and only if $b - c < a < b + c$.

**(c)** Show $|a - b| \leq c$ if and only if $b - c \leq a \leq b + c$.

3.8 Let $a, b \in \mathbb{R}$. Show if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.

Since $b - a > 0$, the Archimedean property shows there exists an $n \in \mathbb{N}$ such that

$$n(b-a) > 1, \quad \text{and hence} \quad bn - an > 1. \tag{2}$$

From this, it is fairly evident that there is an integer $m$ between $an$ and $bn$, so that (1) holds. However, the proof that such an $m$ exists is a little delicate. We argue as follows. By the Archimedean property again, there exists an integer $k > \max\{|an|, |bn|\}$, so that

$$-k < an < bn < k.$$

Then the sets $K = \{j \in \mathbb{Z} : -k \le j \le k\}$ and $\{j \in K : an < j\}$ are finite, and they are nonempty, since they both contain $k$. Let $m = \min\{j \in K : an < j\}$. Then $-k < an < m$. Since $m > -k$, we have $m - 1$ in $K$, so the inequality $an < m - 1$ is false by our choice of $m$. Thus $m - 1 \le an$ and, using (2), we have $m \le an + 1 < bn$. Since $an < m < bn$, (1) holds. ∎

## Exercises

4.1 For each set below that is bounded above, list three upper bounds for the set.[2] Otherwise write "NOT BOUNDED ABOVE" or "NBA."

**(a)** $[0, 1]$
**(b)** $(0, 1)$
**(c)** $\{2, 7\}$
**(d)** $\{\pi, e\}$
**(e)** $\{\frac{1}{n} : n \in \mathbb{N}\}$
**(f)** $\{0\}$
**(g)** $[0, 1] \cup [2, 3]$
**(h)** $\cup_{n=1}^{\infty} [2n, 2n+1]$
**(i)** $\cap_{n=1}^{\infty} [-\frac{1}{n}, 1 + \frac{1}{n}]$
**(j)** $\{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$
**(k)** $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$
**(l)** $\{r \in \mathbb{Q} : r < 2\}$
**(m)** $\{r \in \mathbb{Q} : r^2 < 4\}$
**(n)** $\{r \in \mathbb{Q} : r^2 < 2\}$
**(o)** $\{x \in \mathbb{R} : x < 0\}$
**(p)** $\{1, \frac{\pi}{3}, \pi^2, 10\}$
**(q)** $\{0, 1, 2, 4, 8, 16\}$
**(r)** $\cap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n})$
**(s)** $\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\}$
**(t)** $\{x \in \mathbb{R} : x^3 < 8\}$
**(u)** $\{x^2 : x \in \mathbb{R}\}$
**(v)** $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$
**(w)** $\{\sin(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

4.2 Repeat Exercise 4.1 for lower bounds.

4.3 For each set in Exercise 4.1, give its supremum if it has one. Otherwise write "NO sup."

[2] An integer $p \ge 2$ is a *prime* provided the only positive factors of $p$ are 1 and $p$.

4.4 Repeat Exercise 4.3 for infima [plural of infimum].

4.5 Let $S$ be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove if $\sup S$ belongs to $S$, then $\sup S = \max S$. *Hint*: Your proof should be very short.

4.6 Let $S$ be a nonempty bounded subset of $\mathbb{R}$.

**(a)** Prove $\inf S \leq \sup S$. *Hint*: This is almost obvious; your proof should be short.

**(b)** What can you say about $S$ if $\inf S = \sup S$?

4.7 Let $S$ and $T$ be nonempty bounded subsets of $\mathbb{R}$.

**(a)** Prove if $S \subseteq T$, then $\inf T \leq \inf S \leq \sup S \leq \sup T$.

**(b)** Prove $\sup(S \cup T) = \max\{\sup S, \sup T\}$. *Note*: In part (b), do *not* assume $S \subseteq T$.

4.8 Let $S$ and $T$ be nonempty subsets of $\mathbb{R}$ with the following property: $s \leq t$ for all $s \in S$ and $t \in T$.

**(a)** Observe $S$ is bounded above and $T$ is bounded below.

**(b)** Prove $\sup S \leq \inf T$.

**(c)** Give an example of such sets $S$ and $T$ where $S \cap T$ is nonempty.

**(d)** Give an example of sets $S$ and $T$ where $\sup S = \inf T$ and $S \cap T$ is the empty set.

4.9 Complete the proof that $\inf S = -\sup(-S)$ in Corollary 4.5 by proving (1) and (2).

4.10 Prove that if $a > 0$, then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

4.11 Consider $a, b \in \mathbb{R}$ where $a < b$. Use Denseness of $\mathbb{Q}$ 4.7 to show there are infinitely many rationals between $a$ and $b$.

4.12 Let $\mathbb{I}$ be the set of real numbers that are not rational; elements of $\mathbb{I}$ are called *irrational numbers*. Prove if $a < b$, then there exists $x \in \mathbb{I}$ such that $a < x < b$. *Hint*: First show $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$.

4.13 Prove the following are equivalent for real numbers $a, b, c$. [*Equivalent* means that either all the properties hold or none of the properties hold.]

**(i)** $|a - b| < c$,

**(ii)** $b - c < a < b + c$,

**(iii)** $a \in (b-c, b+c)$.

*Hint*: Use Exercise 3.7(b).

4.14 Let $A$ and $B$ be nonempty bounded subsets of $\mathbb{R}$, and let $A+B$ be the set of all sums $a+b$ where $a \in A$ and $b \in B$.

**(a)** Prove $\sup(A+B) = \sup A + \sup B$. *Hint*: To show $\sup A + \sup B \leq \sup(A+B)$, show that for each $b \in B$, $\sup(A+B) - b$ is an upper bound for $A$, hence $\sup A \leq \sup(A+B) - b$. Then show $\sup(A+B) - \sup A$ is an upper bound for $B$.

**(b)** Prove $\inf(A+B) = \inf A + \inf B$.

4.15 Let $a, b \in \mathbb{R}$. Show if $a \leq b + \frac{1}{n}$ for all $n \in \mathbb{N}$, then $a \leq b$. Compare Exercise 3.8.

4.16 Show $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

# §5 The Symbols $+\infty$ and $-\infty$

The symbols $+\infty$ and $-\infty$ are extremely useful even though they are *not* real numbers. We will often write $+\infty$ as simply $\infty$. We will adjoin $+\infty$ and $-\infty$ to the set $\mathbb{R}$ and extend our ordering to the set $\mathbb{R} \cup \{-\infty, +\infty\}$. Explicitly, we will agree that $-\infty \leq a \leq +\infty$ for all $a$ in $\mathbb{R} \cup \{-\infty, \infty\}$. This provides the set $\mathbb{R} \cup \{-\infty, +\infty\}$ with an ordering that satisfies properties O1, O2 and O3 of §3. We emphasize we will *not* provide the set $\mathbb{R} \cup \{-\infty, +\infty\}$ with any algebraic structure. We may use the symbols $+\infty$ and $-\infty$, but we must continue to remember they do not represent real numbers. Do *not* apply a theorem or exercise that is stated for real numbers to the symbols $+\infty$ or $-\infty$.

It is convenient to use the symbols $+\infty$ and $-\infty$ to extend the notation established in Example 1(b) of §4 to unbounded intervals. For real numbers $a, b \in \mathbb{R}$, we adopt the following

$$[a, \infty) = \{x \in \mathbb{R} : a \leq x\}, \qquad (a, \infty) = \{x \in \mathbb{R} : a < x\},$$
$$(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}, \qquad (-\infty, b) = \{x \in \mathbb{R} : x < b\}.$$

We occasionally also write $(-\infty, \infty)$ for $\mathbb{R}$. $[a, \infty)$ and $(-\infty, b]$ are called *closed intervals* or *unbounded closed intervals*, while $(a, \infty)$ and

## Exercises

5.1 Write the following sets in interval notation:
**(a)** $\{x \in \mathbb{R} : x < 0\}$ **(b)** $\{x \in \mathbb{R} : x^3 \leq 8\}$
**(c)** $\{x^2 : x \in \mathbb{R}\}$ **(d)** $\{x \in \mathbb{R} : x^2 < 8\}$

5.2 Give the infimum and supremum of each set listed in Exercise 5.1.

5.3 Give the infimum and supremum of each unbounded set listed in Exercise 4.1.

5.4 Let $S$ be a nonempty subset of $\mathbb{R}$, and let $-S = \{-s : s \in S\}$. Prove $\inf S = -\sup(-S)$. *Hint*: For the case $-\infty < \inf S$, simply state that this was proved in Exercise 4.9.

5.5 Prove $\inf S \leq \sup S$ for every nonempty subset of $\mathbb{R}$. Compare Exercise 4.6(a).

5.6 Let $S$ and $T$ be nonempty subsets of $\mathbb{R}$ such that $S \subseteq T$. Prove $\inf T \leq \inf S \leq \sup S \leq \sup T$. Compare Exercise 4.7(a).

5.7 Finish Example 1 by verifying the equality involving infimums.

# §6 * A Development of $\mathbb{R}$

There are several ways to give a careful development of $\mathbb{R}$ based on $\mathbb{Q}$. We will briefly discuss one of them and give suggestions for further reading on this topic. [See the remarks about enrichment sections in the preface.]

To motivate our development we begin by observing

$$a = \sup\{r \in \mathbb{Q} : r < a\} \quad \text{for each} \quad a \in \mathbb{R};$$

see Exercise 4.16. Note the intimate relationship: $a \leq b$ if and only if $\{r \in \mathbb{Q} : r < a\} \subseteq \{r \in \mathbb{Q} : r < b\}$ and, moreover, $a = b$ if and only if $\{r \in \mathbb{Q} : r < a\} = \{r \in \mathbb{Q} : r < b\}$. Subsets $\alpha$ of $\mathbb{Q}$ having the form $\{r \in \mathbb{Q} : r < a\}$ satisfy these properties:

**(i)** $\alpha \neq \mathbb{Q}$ and $\alpha$ is not empty,
**(ii)** If $r \in \alpha$, $s \in \mathbb{Q}$ and $s < r$, then $s \in \alpha$,
**(iii)** $\alpha$ contains no largest rational.

## Exercises

6.1 Consider $s, t \in \mathbb{Q}$. Show

**(a)** $s \le t$ if and only if $s^* \subseteq t^*$;

**(b)** $s = t$ if and only if $s^* = t^*$;

**(c)** $(s+t)^* = s^* + t^*$. Note that $s^* + t^*$ is a sum of Dedekind cuts.

6.2 Show that if $\alpha$ and $\beta$ are Dedekind cuts, then so is $\alpha + \beta = \{r_1 + r_2 : r_1 \in \alpha \text{ and } r_2 \in \beta\}$.

6.3 **(a)** Show $\alpha + 0^* = \alpha$ for all Dedekind cuts $\alpha$.

**(b)** We claimed, without proof, that addition of Dedekind cuts satisfies property A4. Thus if $\alpha$ is a Dedekind cut, there is a Dedekind cut $-\alpha$ such that $\alpha + (-\alpha) = 0^*$. How would you define $-\alpha$?

6.4 Let $\alpha$ and $\beta$ be Dedekind cuts and define the "product": $\alpha \cdot \beta = \{r_1 r_2 : r_1 \in \alpha \text{ and } r_2 \in \beta\}$.

**(a)** Calculate some "products" of Dedekind cuts using the Dedekind cuts $0^*$, $1^*$ and $(-1)^*$.

**(b)** Discuss why this definition of "product" is totally unsatisfactory for defining multiplication in $\mathbb{R}$.

6.5 **(a)** Show $\{r \in \mathbb{Q} : r^3 < 2\}$ is a Dedekind cut, but $\{r \in \mathbb{Q} : r^2 < 2\}$ is not a Dedekind cut.

**(b)** Does the Dedekind cut $\{r \in \mathbb{Q} : r^3 < 2\}$ correspond to a rational number in $\mathbb{R}$?

**(c)** Show $0^* \cup \{r \in \mathbb{Q} : r \ge 0 \text{ and } r^2 < 2\}$ is a Dedekind cut. Does it correspond to a rational number in $\mathbb{R}$?

This shows $|s-t| < \epsilon$ for all $\epsilon > 0$. It follows that $|s-t| = 0$; hence $s = t$.

## Exercises

7.1 Write out the first five terms of the following sequences.

**(a)** $s_n = \frac{1}{3n+1}$ **(b)** $b_n = \frac{3n+1}{4n-1}$
**(c)** $c_n = \frac{n}{3^n}$ **(d)** $\sin(\frac{n\pi}{4})$

7.2 For each sequence in Exercise 7.1, determine whether it converges. If it converges, give its limit. No proofs are required.

7.3 For each sequence below, determine whether it converges and, if it converges, give its limit. No proofs are required.

**(a)** $a_n = \frac{n}{n+1}$ **(b)** $b_n = \frac{n^2+3}{n^2-3}$
**(c)** $c_n = 2^{-n}$ **(d)** $t_n = 1 + \frac{2}{n}$
**(e)** $x_n = 73 + (-1)^n$ **(f)** $s_n = (2)^{1/n}$
**(g)** $y_n = n!$ **(h)** $d_n = (-1)^n n$
**(i)** $\frac{(-1)^n}{n}$ **(j)** $\frac{7n^3+8n}{2n^3-3}$
**(k)** $\frac{9n^2-18}{6n+18}$ **(l)** $\sin(\frac{n\pi}{2})$
**(m)** $\sin(n\pi)$ **(n)** $\sin(\frac{2n\pi}{3})$
**(o)** $\frac{1}{n}\sin n$ **(p)** $\frac{2^{n+1}+5}{2^n-7}$
**(q)** $\frac{3^n}{n!}$ **(r)** $(1+\frac{1}{n})^2$
**(s)** $\frac{4n^2+3}{3n^2-2}$ **(t)** $\frac{6n+4}{9n^2+7}$

7.4 Give examples of

**(a)** A sequence $(x_n)$ of irrational numbers having a limit $\lim x_n$ that is a rational number.

**(b)** A sequence $(r_n)$ of rational numbers having a limit $\lim r_n$ that is an irrational number.

7.5 Determine the following limits. No proofs are required, but show any relevant algebra.

**(a)** $\lim s_n$ where $s_n = \sqrt{n^2+1} - n$,

**(b)** $\lim(\sqrt{n^2+n} - n)$,

**(c)** $\lim(\sqrt{4n^2+n} - 2n)$.
*Hint for* (a): First show $s_n = \frac{1}{\sqrt{n^2+1}+n}$.

then we clearly have $m > 0$ and $|s_n| \geq m$ for *all* $n \in \mathbb{N}$ in view of (1). Thus $\inf\{|s_n| : n \in \mathbb{N}\} \geq m > 0$, as desired. ■

Formal proofs are required in the following exercises.

## Exercises

8.1 Prove the following:

**(a)** $\lim \frac{(-1)^n}{n} = 0$ **(b)** $\lim \frac{1}{n^{1/3}} = 0$

**(c)** $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$ **(d)** $\lim \frac{n+6}{n^2-6} = 0$

8.2 Determine the limits of the following sequences, and then prove your claims.

**(a)** $a_n = \frac{n}{n^2+1}$ **(b)** $b_n = \frac{7n-19}{3n+7}$

**(c)** $c_n = \frac{4n+3}{7n-5}$ **(d)** $d_n = \frac{2n+4}{5n+2}$

**(e)** $s_n = \frac{1}{n}\sin n$

8.3 Let $(s_n)$ be a sequence of nonnegative real numbers, and suppose $\lim s_n = 0$. Prove $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

8.4 Let $(t_n)$ be a bounded sequence, i.e., there exists $M$ such that $|t_n| \leq M$ for all $n$, and let $(s_n)$ be a sequence such that $\lim s_n = 0$. Prove $\lim(s_n t_n) = 0$.

8.5 ★[1]

**(a)** Consider three sequences $(a_n)$, $(b_n)$ and $(s_n)$ such that $a_n \leq s_n \leq b_n$ for all $n$ and $\lim a_n = \lim b_n = s$. Prove $\lim s_n = s$. This is called the "squeeze lemma."

**(b)** Suppose $(s_n)$ and $(t_n)$ are sequences such that $|s_n| \leq t_n$ for all $n$ and $\lim t_n = 0$. Prove $\lim s_n = 0$.

8.6 Let $(s_n)$ be a sequence in $\mathbb{R}$.

**(a)** Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

**(b)** Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

8.7 Show the following sequences do not converge.

**(a)** $\cos(\frac{n\pi}{3})$ **(b)** $s_n = (-1)^n n$

**(c)** $\sin(\frac{n\pi}{3})$

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[1]This exercise is referred to in several places.

8.8 Prove the following [see Exercise 7.5]:
**(a)** $\lim[\sqrt{n^2+1}-n]=0$ **(b)** $\lim[\sqrt{n^2+n}-n]=\frac{1}{2}$
**(c)** $\lim[\sqrt{4n^2+n}-2n]=\frac{1}{4}$

8.9 ★[2] Let $(s_n)$ be a sequence that converges.

**(a)** Show that if $s_n \geq a$ for all but finitely many $n$, then $\lim s_n \geq a$.

**(b)** Show that if $s_n \leq b$ for all but finitely many $n$, then $\lim s_n \leq b$.

**(c)** Conclude that if all but finitely many $s_n$ belong to $[a,b]$, then $\lim s_n$ belongs to $[a,b]$.

8.10 Let $(s_n)$ be a convergent sequence, and suppose $\lim s_n > a$. Prove there exists a number $N$ such that $n > N$ implies $s_n > a$.

# §9 Limit Theorems for Sequences

In this section we prove some basic results that are probably already familiar to the reader. First we prove convergent sequences are bounded. A sequence $(s_n)$ of real numbers is said to be *bounded* if the set $\{s_n : n \in \mathbb{N}\}$ is a bounded set, i.e., if there exists a constant $M$ such that $|s_n| \leq M$ for all $n$.

**9.1 Theorem.**
*Convergent sequences are bounded.*

**Proof**
Let $(s_n)$ be a convergent sequence, and let $s = \lim s_n$. Applying Definition 7.1 with $\epsilon = 1$ we obtain $N$ in $\mathbb{N}$ so that

$$n > N \quad \text{implies} \quad |s_n - s| < 1.$$

From the triangle inequality we see $n > N$ implies $|s_n| < |s| + 1$. Define $M = \max\{|s|+1, |s_1|, |s_2|, \ldots, |s_N|\}$. Then we have $|s_n| \leq M$ for all $n \in \mathbb{N}$, so $(s_n)$ is a bounded sequence. ■

In the proof of Theorem 9.1 we only needed to use property 7.1(1) for a single value of $\epsilon$. Our choice of $\epsilon = 1$ was quite arbitrary.

---

[2]This exercise is referred to in several places.

$\epsilon > 0$, so there exists $N$ such that $n > N$ implies $|\frac{1}{s_n} - 0| < \epsilon = \frac{1}{M}$. Since $s_n > 0$, we can write

$$n > N \quad \text{implies} \quad 0 < \frac{1}{s_n} < \frac{1}{M}$$

and hence

$$n > N \quad \text{implies} \quad M < s_n.$$

That is, $\lim s_n = +\infty$ and (2) holds. ■

# Exercises

9.1 Using the limit Theorems 9.2–9.7, prove the following. Justify all steps.
**(a)** $\lim \frac{n+1}{n} = 1$ **(b)** $\lim \frac{3n+7}{6n-5} = \frac{1}{2}$
**(c)** $\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} = \frac{17}{23}$

9.2 Suppose $\lim x_n = 3$, $\lim y_n = 7$ and all $y_n$ are nonzero. Determine the following limits:
**(a)** $\lim(x_n + y_n)$ **(b)** $\lim \frac{3y_n - x_n}{y_n^2}$

9.3 Suppose $\lim a_n = a$, $\lim b_n = b$, and $s_n = \frac{a_n^3+4a_n}{b_n^2+1}$. Prove $\lim s_n = \frac{a^3+4a}{b^2+1}$ carefully, using the limit theorems.

9.4 Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

**(a)** List the first four terms of $(s_n)$.

**(b)** It turns out that $(s_n)$ converges. Assume this fact and prove the limit is $\frac{1}{2}(1 + \sqrt{5})$.

9.5 Let $t_1 = 1$ and $t_{n+1} = \frac{t_n^2+2}{2t_n}$ for $n \geq 1$. Assume $(t_n)$ converges and find the limit.

9.6 Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.

**(a)** Show if $a = \lim x_n$, then $a = \frac{1}{3}$ or $a = 0$.

**(b)** Does $\lim x_n$ exist? Explain.

**(c)** Discuss the apparent contradiction between parts (a) and (b).

9.7 Complete the proof of Theorem 9.7(c), i.e., give the standard argument needed to show $\lim s_n = 0$.

9.8 Give the following when they exist. Otherwise assert "NOT EXIST."

**(a)** $\lim n^3$ **(b)** $\lim(-n^3)$

**(c)** $\lim(-n)^n$ **(d)** $\lim(1.01)^n$

**(e)** $\lim n^n$

9.9 Suppose there exists $N_0$ such that $s_n \le t_n$ for all $n > N_0$.

**(a)** Prove that if $\lim s_n = +\infty$, then $\lim t_n = +\infty$.

**(b)** Prove that if $\lim t_n = -\infty$, then $\lim s_n = -\infty$.

**(c)** Prove that if $\lim s_n$ and $\lim t_n$ exist, then $\lim s_n \le \lim t_n$.

9.10 **(a)** Show that if $\lim s_n = +\infty$ and $k > 0$, then $\lim(ks_n) = +\infty$.

**(b)** Show $\lim s_n = +\infty$ if and only if $\lim(-s_n) = -\infty$.

**(c)** Show that if $\lim s_n = +\infty$ and $k < 0$, then $\lim(ks_n) = -\infty$.

9.11 **(a)** Show that if $\lim s_n = +\infty$ and $\inf\{t_n : n \in \mathbb{N}\} > -\infty$, then $\lim(s_n + t_n) = +\infty$.

**(b)** Show that if $\lim s_n = +\infty$ and $\lim t_n > -\infty$, then $\lim(s_n + t_n) = +\infty$.

**(c)** Show that if $\lim s_n = +\infty$ and if $(t_n)$ is a bounded sequence, then $\lim(s_n + t_n) = +\infty$.

9.12 ★[3] Assume all $s_n \neq 0$ and that the limit $L = \lim |\frac{s_{n+1}}{s_n}|$ exists.

**(a)** Show that if $L < 1$, then $\lim s_n = 0$. *Hint*: Select $a$ so that $L < a < 1$ and obtain $N$ so that $|s_{n+1}| < a|s_n|$ for $n \ge N$. Then show $|s_n| < a^{n-N}|s_N|$ for $n > N$.

**(b)** Show that if $L > 1$, then $\lim |s_n| = +\infty$. *Hint*: Apply (a) to the sequence $t_n = \frac{1}{|s_n|}$; see Theorem 9.10.

9.13 Show

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ 1 & \text{if } a = 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a \le -1. \end{cases}$$

9.14 Let $p > 0$. Use Exercise 9.12 to show

$$\lim_{n\to\infty} \frac{a^n}{n^p} = \begin{cases} 0 & \text{if } |a| \le 1 \\ +\infty & \text{if } a > 1 \\ \text{does not exist} & \text{if } a < -1. \end{cases}$$

*Hint*: For the $a > 1$ case, use Exercise 9.12(b).

[3] This exercise is referred to in several places.

9.15 Show $\lim_{n\to\infty} \frac{a^n}{n!} = 0$ for all $a \in \mathbb{R}$.

9.16 Use Theorems 9.9 and 9.10 or Exercises 9.9–9.15 to prove the following:

**(a)** $\lim \frac{n^4+8n}{n^2+9} = +\infty$

**(b)** $\lim[\frac{2^n}{n^2} + (-1)^n] = +\infty$

**(c)** $\lim[\frac{3^n}{n^3} - \frac{3^n}{n!}] = +\infty$

9.17 Give a formal proof that $\lim n^2 = +\infty$ using only Definition 9.8.

9.18 **(a)** Verify $1 + a + a^2 + \cdots + a^n = \frac{1-a^{n+1}}{1-a}$ for $a \neq 1$.

**(b)** Find $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $|a| < 1$.

**(c)** Calculate $\lim_{n\to\infty}(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^n})$.

**(d)** What is $\lim_{n\to\infty}(1 + a + a^2 + \cdots + a^n)$ for $a \geq 1$?

# §10 Monotone Sequences and Cauchy Sequences

In this section we obtain two theorems [Theorems 10.2 and 10.11] that will allow us to conclude certain sequences converge *without* knowing the limit in advance. These theorems are important because in practice the limits are not usually known in advance.

**10.1 Definition.**
A sequence $(s_n)$ of real numbers is called an *increasing sequence* if $s_n \leq s_{n+1}$ for all $n$, and $(s_n)$ is called a *decreasing sequence* if $s_n \geq s_{n+1}$ for all $n$. Note that if $(s_n)$ is increasing, then $s_n \leq s_m$ whenever $n < m$. A sequence that is increasing or decreasing[4] will be called a *monotone sequence* or a *monotonic sequence*.

**Example 1**
The sequences defined by $a_n = 1 - \frac{1}{n}$, $b_n = n^3$ and $c_n = (1+\frac{1}{n})^n$ are increasing sequences, although this is not obvious for the

[4]In the First Edition of this book, increasing and decreasing sequences were referred to as “nondecreasing” and “nonincreasing” sequences, respectively.

**Proof**

The expression "if and only if" indicates that we have two assertions to verify: (i) convergent sequences are Cauchy sequences, and (ii) Cauchy sequences are convergent sequences. We already verified (i) in Lemma 10.9. To check (ii), consider a Cauchy sequence $(s_n)$ and note $(s_n)$ is bounded by Lemma 10.10. By Theorem 10.7 we need only show

$$\liminf s_n = \limsup s_n. \tag{1}$$

Let $\epsilon > 0$. Since $(s_n)$ is a Cauchy sequence, there exists $N$ so that

$$m, n > N \quad \text{implies} \quad |s_n - s_m| < \epsilon.$$

In particular, $s_n < s_m + \epsilon$ for all $m, n > N$. This shows $s_m + \epsilon$ is an upper bound for $\{s_n : n > N\}$, so $v_N = \sup\{s_n : n > N\} \le s_m + \epsilon$ for $m > N$. This, in turn, shows $v_N - \epsilon$ is a lower bound for $\{s_m : m > N\}$, so $v_N - \epsilon \le \inf\{s_m : m > N\} = u_N$. Thus

$$\limsup s_n \le v_N \le u_N + \epsilon \le \liminf s_n + \epsilon.$$

Since this holds for all $\epsilon > 0$, we have $\limsup s_n \le \liminf s_n$. The opposite inequality always holds, so we have established (1). ■

The proof of Theorem 10.11 uses Theorem 10.7, and Theorem 10.7 relies implicitly on the Completeness Axiom 4.4, since without the completeness axiom it is not clear that $\liminf s_n$ and $\limsup s_n$ are meaningful. The completeness axiom assures us that the expressions $\sup\{s_n : n > N\}$ and $\inf\{s_n : n > N\}$ in Definition 10.6 are meaningful, and Theorem 10.2 [which itself relies on the completeness axiom] assures us that the limits in Definition 10.6 also are meaningful.

Exercises on lim sup's and lim inf's appear in §§11 and 12.

# Exercises

10.1 Which of the following sequences are increasing? decreasing? bounded?

**(a)** $\frac{1}{n}$ **(b)** $\frac{(-1)^n}{n^2}$

**(c)** $n^5$ **(d)** $\sin(\frac{n\pi}{7})$

**(e)** $(-2)^n$ **(f)** $\frac{n}{3^n}$

10.2 Prove Theorem 10.2 for bounded decreasing sequences.

10.3 For a decimal expansion $K.d_1d_2d_3d_4\cdots$, let $(s_n)$ be defined as in Discussion 10.3. Prove $s_n < K+1$ for all $n \in \mathbb{N}$. *Hint*: $\frac{9}{10} + \frac{9}{10^2} + \cdots + \frac{9}{10^n} = 1 - \frac{1}{10^n}$ for all $n$.

10.4 Discuss why Theorems 10.2 and 10.11 would fail if we restricted our world of numbers to the set $\mathbb{Q}$ of rational numbers.

10.5 Prove Theorem 10.4(ii).

10.6 **(a)** Let $(s_n)$ be a sequence such that

$$|s_{n+1} - s_n| < 2^{-n} \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove $(s_n)$ is a Cauchy sequence and hence a convergent sequence.

**(b)** Is the result in (a) true if we only assume $|s_{n+1} - s_n| < \frac{1}{n}$ for all $n \in \mathbb{N}$?

10.7 Let $S$ be a bounded nonempty subset of $\mathbb{R}$ such that $\sup S$ is not in $S$. Prove there is a sequence $(s_n)$ of points in $S$ such that $\lim s_n = \sup S$. See also Exercise 11.11.

10.8 Let $(s_n)$ be an increasing sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove $(\sigma_n)$ is an increasing sequence.

10.9 Let $s_1 = 1$ and $s_{n+1} = (\frac{n}{n+1})s_n^2$ for $n \geq 1$.

**(a)** Find $s_2$, $s_3$ and $s_4$.

**(b)** Show $\lim s_n$ exists.

**(c)** Prove $\lim s_n = 0$.

10.10 Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.

**(a)** Find $s_2$, $s_3$ and $s_4$.

**(b)** Use induction to show $s_n > \frac{1}{2}$ for all $n$.

**(c)** Show $(s_n)$ is a decreasing sequence.

**(d)** Show $\lim s_n$ exists and find $\lim s_n$.

10.11 Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{4n^2}] \cdot t_n$ for $n \geq 1$.

**(a)** Show $\lim t_n$ exists.

**(b)** What do you think $\lim t_n$ is?

10.12 Let $t_1 = 1$ and $t_{n+1} = [1 - \frac{1}{(n+1)^2}] \cdot t_n$ for $n \geq 1$.

**(a)** Show $\lim t_n$ exists.

**(b)** What do you think $\lim t_n$ is?

**(c)** Use induction to show $t_n = \frac{n+1}{2n}$.

**(d)** Repeat part (b).

# §11 Subsequences

**11.1 Definition.**
Suppose $(s_n)_{n\in\mathbb{N}}$ is a sequence. A *subsequence* of this sequence is a sequence of the form $(t_k)_{k\in\mathbb{N}}$ where for each $k$ there is a positive integer $n_k$ such that

$$n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots \tag{1}$$

and

$$t_k = s_{n_k}. \tag{2}$$

Thus $(t_k)$ is just a selection of some [possibly all] of the $s_n$'s taken in order.

Here are some alternative ways to approach this concept. Note that (1) defines an infinite subset of $\mathbb{N}$, namely $\{n_1, n_2, n_3, \ldots\}$. Conversely, every infinite subset of $\mathbb{N}$ can be described by (1). Thus a subsequence of $(s_n)$ is a sequence obtained by selecting, in order, an infinite subset of the terms.

For a more precise definition, recall we can view the sequence $(s_n)_{n\in\mathbb{N}}$ as a function $s$ with domain $\mathbb{N}$; see §7. For the subset $\{n_1, n_2, n_3, \ldots\}$, there is a natural function $\sigma$ [lower case Greek sigma] given by $\sigma(k) = n_k$ for $k \in \mathbb{N}$. The function $\sigma$ "selects" an infinite subset of $\mathbb{N}$, in order. The subsequence of $s$ corresponding to $\sigma$ is simply the composite function $t = s \circ \sigma$. That is,

$$t_k = t(k) = s \circ \sigma(k) = s(\sigma(k)) = s(n_k) = s_{n_k} \quad \text{for} \quad k \in \mathbb{N}. \tag{3}$$

Thus a sequence $t$ is a subsequence of a sequence $s$ if and only if $t = s \circ \sigma$ for some increasing function $\sigma$ mapping $\mathbb{N}$ into $\mathbb{N}$. We will usually suppress the notation $\sigma$ and often suppress the notation $t$

**Proof**

Suppose $t$ is finite. Consider the interval $(t-\epsilon, t+\epsilon)$. Then some $t_n$ is in this interval. Let $\delta = \min\{t+\epsilon-t_n, t_n-t+\epsilon\}$, so that

$$(t_n-\delta, t_n+\delta) \subseteq (t-\epsilon, t+\epsilon).$$

Since $t_n$ is a subsequential limit, the set $\{n \in \mathbb{N} : s_n \in (t_n-\delta, t_n+\delta)\}$ is infinite, so the set $\{n \in \mathbb{N} : s_n \in (t-\epsilon, t+\epsilon)\}$ is also infinite. Thus, by Theorem 11.2(i), $t$ itself is a subsequential limit of $(s_n)$.

If $t = +\infty$, then clearly the sequence $(s_n)$ is unbounded above, so a subsequence of $(s_n)$ has limit $+\infty$ by Theorem 11.2(ii). Thus $+\infty$ is also in $S$. A similar argument applies if $t = -\infty$. ∎

## Exercises

11.1 Let $a_n = 3 + 2(-1)^n$ for $n \in \mathbb{N}$.

- **(a)** List the first eight terms of the sequence $(a_n)$.
- **(b)** Give a subsequence that is constant [takes a single value]. Specify the selection function $\sigma$.

11.2 Consider the sequences defined as follows:

$$a_n = (-1)^n, \qquad b_n = \frac{1}{n}, \qquad c_n = n^2, \qquad d_n = \frac{6n+4}{7n-3}.$$

- **(a)** For each sequence, give an example of a monotone subsequence.
- **(b)** For each sequence, give its set of subsequential limits.
- **(c)** For each sequence, give its $\limsup$ and $\liminf$.
- **(d)** Which of the sequences converges? diverges to $+\infty$? diverges to $-\infty$?
- **(e)** Which of the sequences is bounded?

11.3 Repeat Exercise 11.2 for the sequences:

$$s_n = \cos(\frac{n\pi}{3}), \quad t_n = \frac{3}{4n+1}, \quad u_n = \left(-\frac{1}{2}\right)^n, \quad v_n = (-1)^n + \frac{1}{n}.$$

11.4 Repeat Exercise 11.2 for the sequences:

$$w_n = (-2)^n, \quad x_n = 5^{(-1)^n}, \quad y_n = 1 + (-1)^n, \quad z_n = n\cos\left(\frac{n\pi}{4}\right).$$

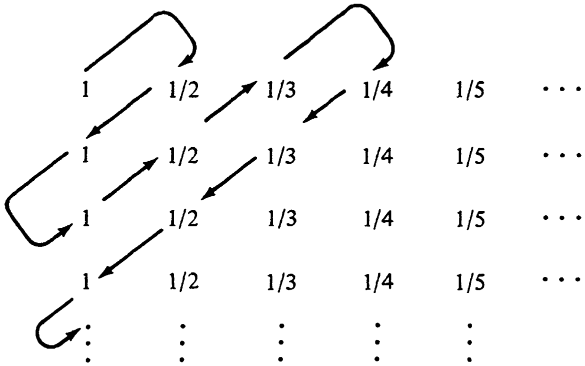


**FIGURE 11.2**

11.5 Let $(q_n)$ be an enumeration of all the rationals in the interval $(0, 1]$.

**(a)** Give the set of subsequential limits for $(q_n)$.

**(b)** Give the values of $\limsup q_n$ and $\liminf q_n$.

11.6 Show every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence. *Hint*: Define subsequences as in (3) of Definition 11.1.

11.7 Let $(r_n)$ be an enumeration of the set $\mathbb{Q}$ of all rational numbers. Show there exists a subsequence $(r_{n_k})$ such that $\lim_{k\to\infty} r_{n_k} = +\infty$.

11.8 ★[8] Use Definition 10.6 and Exercise 5.4 to prove $\liminf s_n = -\limsup(-s_n)$ for every sequence $(s_n)$.

11.9 **(a)** Show the closed interval $[a, b]$ is a closed set.

**(b)** Is there a sequence $(s_n)$ such that $(0, 1)$ is its set of subsequential limits?

11.10 Let $(s_n)$ be the sequence of numbers in Fig. 11.2 listed in the indicated order.

**(a)** Find the set $S$ of subsequential limits of $(s_n)$.

**(b)** Determine $\limsup s_n$ and $\liminf s_n$.

[8] This exercise is referred to in several places.

11.11 Let $S$ be a bounded set. Prove there is an increasing sequence $(s_n)$ of points in $S$ such that $\lim s_n = \sup S$. Compare Exercise 10.7. *Note*: If $\sup S$ is in $S$, it's sufficient to define $s_n = \sup S$ for all $n$.

# §12 lim sup's and lim inf's

Let $(s_n)$ be any sequence of real numbers, and let $S$ be the set of subsequential limits of $(s_n)$. Recall

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} = \sup S \tag{*}$$

and

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} = \inf S. \tag{**}$$

The first equalities in (*) and (**) are from Definition 10.6, and the second equalities are proved in Theorem 11.8. This section is designed to increase the students' familiarity with these concepts. Most of the material is given in the exercises. We illustrate the techniques by proving some results that will be needed later in the text.

**12.1 Theorem.**
*If $(s_n)$ converges to a positive real number $s$ and $(t_n)$ is any sequence, then*

$$\limsup s_n t_n = s \cdot \limsup t_n.$$

*Here we allow the conventions $s \cdot (+\infty) = +\infty$ and $s \cdot (-\infty) = -\infty$ for $s > 0$.*

**Proof**
We first show

$$\limsup s_n t_n \geq s \cdot \limsup t_n. \tag{1}$$

We have three cases. Let $\beta = \limsup t_n$.

*Case* 1. Suppose $\beta$ is finite.

By Theorem 11.7, there exists a subsequence $(t_{n_k})$ of $(t_n)$ such that $\lim_{k\to\infty} t_{n_k} = \beta$. We also have $\lim_{k\to\infty} s_{n_k} = s$ [by Theorem 11.3], so $\lim_{k\to\infty} s_{n_k} t_{n_k} = s\beta$. Thus $(s_{n_k} t_{n_k})$ is a subsequence of $(s_n t_n)$ converging to $s\beta$, and therefore $s\beta \leq \limsup s_n t_n$. [Recall that

## Exercises

12.1 Let $(s_n)$ and $(t_n)$ be sequences and suppose there exists $N_0$ such that $s_n \leq t_n$ for all $n > N_0$. Show $\liminf s_n \leq \liminf t_n$ and $\limsup s_n \leq \limsup t_n$. *Hint*: Use Definition 10.6 and Exercise 9.9(c).

12.2 Prove $\limsup |s_n| = 0$ if and only if $\lim s_n = 0$.

12.3 Let $(s_n)$ and $(t_n)$ be the following sequences that repeat in cycles of four:

$$(s_n) = (0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, 1, 2, 1, 0, \ldots)$$
$$(t_n) = (2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, 1, 1, 0, 2, \ldots)$$

Find
**(a)** $\liminf s_n + \liminf t_n$, **(b)** $\liminf(s_n + t_n)$,
**(c)** $\liminf s_n + \limsup t_n$, **(d)** $\limsup(s_n + t_n)$,
**(e)** $\limsup s_n + \limsup t_n$, **(f)** $\liminf(s_n t_n)$,
**(g)** $\limsup(s_n t_n)$

12.4 Show $\limsup(s_n + t_n) \leq \limsup s_n + \limsup t_n$ for bounded sequences $(s_n)$ and $(t_n)$. *Hint*: First show

$$\sup\{s_n + t_n : n > N\} \leq \sup\{s_n : n > N\} + \sup\{t_n : n > N\}.$$

Then apply Exercise 9.9(c).

12.5 Use Exercises 11.8 and 12.4 to prove

$$\liminf(s_n + t_n) \geq \liminf s_n + \liminf t_n$$

for bounded sequences $(s_n)$ and $(t_n)$.

12.6 Let $(s_n)$ be a bounded sequence, and let $k$ be a nonnegative real number.

**(a)** Prove $\limsup(ks_n) = k \cdot \limsup s_n$.

**(b)** Do the same for $\liminf$. *Hint*: Use Exercise 11.8.

**(c)** What happens in (a) and (b) if $k < 0$?

12.7 Prove if $\limsup s_n = +\infty$ and $k > 0$, then $\limsup(ks_n) = +\infty$.

12.8 Let $(s_n)$ and $(t_n)$ be bounded sequences of nonnegative numbers. Prove $\limsup s_n t_n \leq (\limsup s_n)(\limsup t_n)$.

12.9 **(a)** Prove that if $\lim s_n = +\infty$ and $\liminf t_n > 0$, then $\lim s_n t_n = +\infty$.

**(b)** Prove that if $\limsup s_n = +\infty$ and $\liminf t_n > 0$, then $\limsup s_n t_n = +\infty$.

**(c)** Observe that Exercise 12.7 is the special case of (b) where $t_n = k$ for all $n \in \mathbb{N}$.

12.10 Prove $(s_n)$ is bounded if and only if $\limsup |s_n| < +\infty$.

12.11 Prove the first inequality in Theorem 12.2.

12.12 Let $(s_n)$ be a sequence of nonnegative numbers, and for each $n$ define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$.

**(a)** Show

$$\liminf s_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup s_n.$$

*Hint*: For the last inequality, show first that $M > N$ implies

$$\sup\{\sigma_n : n > M\} \le \frac{1}{M}(s_1 + s_2 + \cdots + s_N) + \sup\{s_n : n > N\}.$$

**(b)** Show that if $\lim s_n$ exists, then $\lim \sigma_n$ exists and $\lim \sigma_n = \lim s_n$.

**(c)** Give an example where $\lim \sigma_n$ exists, but $\lim s_n$ does not exist.

12.13 Let $(s_n)$ be a bounded sequence in $\mathbb{R}$. Let $A$ be the set of $a \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n < a\}$ is finite, i.e., all but finitely many $s_n$ are $\ge a$. Let $B$ be the set of $b \in \mathbb{R}$ such that $\{n \in \mathbb{N} : s_n > b\}$ is finite. Prove $\sup A = \liminf s_n$ and $\inf B = \limsup s_n$.

12.14 Calculate (a) $\lim (n!)^{1/n}$, (b) $\lim \frac{1}{n}(n!)^{1/n}$.

# §13 * Some Topological Concepts in Metric Spaces

In this book we are restricting our attention to analysis on $\mathbb{R}$. Accordingly, we have taken full advantage of the order properties of $\mathbb{R}$ and studied such important notions as lim sup's and lim inf's. In §3 we briefly introduced a distance function on $\mathbb{R}$. Most of our analysis could have been based on the notion of distance, in which case it becomes easy and natural to work in a more general setting. For example, analysis on the $k$-dimensional Euclidean spaces $\mathbb{R}^k$ is important, but these spaces do not have the useful natural ordering that $\mathbb{R}$ has, unless of course $k = 1$.

$y \in E$, we have $d(y,x) < \delta$. Moreover, $d(y,x_k) < \frac{r_k}{2}$ for some $k \in \{1,2,\ldots,n\}$. Therefore, for this $k$ we have

$$d(x,x_k) \le d(x,y) + d(y,x_k) < \delta + \frac{r_k}{2} \le \frac{r_k}{2} + \frac{r_k}{2} = r_k.$$

Thus, by (2) applied to $x = x_k$, we see that $x$ belongs to $U$. Hence (1) holds. □

## Exercises

13.1 For points $\boldsymbol{x}$, $\boldsymbol{y}$ in $\mathbb{R}^k$, let

$$d_1(\boldsymbol{x},\boldsymbol{y}) = \max\{|x_j - y_j| : j = 1,2,\ldots,k\}$$

and

$$d_2(\boldsymbol{x},\boldsymbol{y}) = \sum_{j=1}^{k} |x_j - y_j|.$$

**(a)** Show $d_1$ and $d_2$ are metrics for $\mathbb{R}^k$.

**(b)** Show $d_1$ and $d_2$ are complete metrics on $\mathbb{R}^k$.

13.2 **(a)** Prove (1) in Lemma 13.3.

**(b)** Prove the first assertion in Lemma 13.3.

13.3 Let $B$ be the set of all bounded sequences $\boldsymbol{x} = (x_1, x_2, \ldots)$, and define $d(\boldsymbol{x},\boldsymbol{y}) = \sup\{|x_j - y_j| : j = 1,2,\ldots\}$.

**(a)** Show $d$ is a metric for $B$.

**(b)** Does $d^*(\boldsymbol{x},\boldsymbol{y}) = \sum_{j=1}^{\infty} |x_j - y_j|$ define a metric for $B$?

13.4 Prove (iii) and (iv) in Discussion 13.7.

13.5 **(a)** Verify one of DeMorgan's Laws for sets:

$$\bigcap\{S \setminus U : U \in \mathcal{U}\} = S \setminus \bigcup\{U : U \in \mathcal{U}\}.$$

**(b)** Show that the intersection of any collection of closed sets is a closed set.

13.6 Prove Proposition 13.9.

13.7 Show that every open set in $\mathbb{R}$ is the disjoint union of a finite or infinite sequence of open intervals.

13.8 **(a)** Verify the assertions in the first paragraph of Example 3.

**(b)** Verify the assertions in Example 4.

13.9 Find the closures of the following sets:

**(a)** $\{\frac{1}{n} : n \in \mathbb{N}\}$,

**(b)** $\mathbb{Q}$, the set of rational numbers,

**(c)** $\{r \in \mathbb{Q} : r^2 < 2\}$.

13.10 Show that the interior of each of the following sets is the empty set.

**(a)** $\{\frac{1}{n} : n \in \mathbb{N}\}$,

**(b)** $\mathbb{Q}$, the set of rational numbers,

**(c)** The Cantor set in Example 5.

13.11 Let $E$ be a subset of $\mathbb{R}^k$. Show that $E$ is compact if and only if every sequence in $E$ has a subsequence converging to a point *in* $E$.

13.12 Let $(S, d)$ be any metric space.

**(a)** Show that if $E$ is a closed subset of a compact set $F$, then $E$ is also compact.

**(b)** Show that the finite union of compact sets in $S$ is compact.

13.13 Let $E$ be a compact nonempty subset of $\mathbb{R}$. Show $\sup E$ and $\inf E$ belong to $E$.

13.14 Let $E$ be a compact nonempty subset of $\mathbb{R}^k$, and let $\delta = \sup\{d(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in E\}$. Show $E$ contains points $\boldsymbol{x}_0$, $\boldsymbol{y}_0$ such that $d(\boldsymbol{x}_0, \boldsymbol{y}_0) = \delta$.

13.15 Let $(B, d)$ be as in Exercise 13.3, and let $F$ consist of all $\boldsymbol{x} \in B$ such that $\sup\{|x_j| : j = 1, 2, \ldots\} \le 1$.

**(a)** Show $F$ is closed and bounded. [A set $F$ in a metric space $(S, d)$ is *bounded* if there exist $s_0 \in S$ and $r > 0$ such that $F \subseteq \{s \in S : d(s, s_0) \le r\}$.]

**(b)** Show $F$ is not compact. *Hint*: For each $\boldsymbol{x}$ in $F$, let $U(\boldsymbol{x}) = \{\boldsymbol{y} \in B : d(\boldsymbol{y}, \boldsymbol{x}) < 1\}$, and consider the cover $\mathcal{U}$ of $F$ consisting of all $U(\boldsymbol{x})$. For each $n \in \mathbb{N}$, let $\boldsymbol{x}^{(n)}$ be defined so that $x_n^{(n)} = -1$ and $x_j^{(n)} = 1$ for $j \neq n$. Show that distinct $\boldsymbol{x}^{(n)}$ cannot belong to the same member of $\mathcal{U}$.

## Exercises

14.1 Determine which of the following series converge. Justify your answers.

**(a)** $\sum \frac{n^4}{2^n}$ **(b)** $\sum \frac{2^n}{n!}$
**(c)** $\sum \frac{n^2}{3^n}$ **(d)** $\sum \frac{n!}{n^4+3}$
**(e)** $\sum \frac{\cos^2 n}{n^2}$ **(f)** $\sum_{n=2}^{\infty} \frac{1}{\log n}$

14.2 Repeat Exercise 14.1 for the following.

**(a)** $\sum \frac{n-1}{n^2}$ **(b)** $\sum (-1)^n$
**(c)** $\sum \frac{3n}{n^3}$ **(d)** $\sum \frac{n^3}{3^n}$
**(e)** $\sum \frac{n^2}{n!}$ **(f)** $\sum \frac{1}{n^n}$
**(g)** $\sum \frac{n}{2^n}$

14.3 Repeat Exercise 14.1 for the following.

**(a)** $\sum \frac{1}{\sqrt{n!}}$ **(b)** $\sum \frac{2+\cos n}{3^n}$
**(c)** $\sum \frac{1}{2^n+n}$ **(d)** $\sum (\frac{1}{2})^n (50 + \frac{2}{n})$
**(e)** $\sum \sin(\frac{n\pi}{9})$ **(f)** $\sum \frac{(100)^n}{n!}$

14.4 Repeat Exercise 14.1 for the following.

**(a)** $\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}$ **(b)** $\sum [\sqrt{n+1} - \sqrt{n}]$
**(c)** $\sum \frac{n!}{n^n}$

14.5 Suppose $\sum a_n = A$ and $\sum b_n = B$ where $A$ and $B$ are real numbers. Use limit theorems from §9 to quickly prove the following.

**(a)** $\sum (a_n + b_n) = A + B$.

**(b)** $\sum ka_n = kA$ for $k \in \mathbb{R}$.

**(c)** Is $\sum a_n b_n = AB$ a reasonable conjecture? Discuss.

14.6 **(a)** Prove that if $\sum |a_n|$ converges and $(b_n)$ is a bounded sequence, then $\sum a_n b_n$ converges. *Hint*: Use Theorem 14.4.

**(b)** Observe that Corollary 14.7 is a special case of part (a).

14.7 Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

14.8 Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges. *Hint*: Show $\sqrt{a_n b_n} \le a_n + b_n$ for all $n$.

14.9 The convergence of a series does not depend on any finite number of the terms, though of course the value of the limit does. More precisely, consider series $\sum a_n$ and $\sum b_n$ and suppose the set $\{n \in \mathbb{N} : a_n \neq b_n\}$

is finite. Then the series both converge or else they both diverge. Prove this. *Hint*: This is almost obvious from Theorem 14.4.

14.10 Find a series $\sum a_n$ which diverges by the Root Test but for which the Ratio Test gives no information. Compare Example 8.

14.11 Let $(a_n)$ be a sequence of nonzero real numbers such that the sequence $(\frac{a_{n+1}}{a_n})$ of ratios is a constant sequence. Show $\sum a_n$ is a geometric series.

14.12 Let $(a_n)_{n\in\mathbb{N}}$ be a sequence such that $\liminf |a_n| = 0$. Prove there is a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $\sum_{k=1}^\infty a_{n_k}$ converges.

14.13 We have seen that it is often a lot harder to find the value of an infinite sum than to show it exists. Here are some sums that can be handled.

- **(a)** Calculate $\sum_{n=1}^\infty (\frac{2}{3})^n$ and $\sum_{n=1}^\infty (-\frac{2}{3})^n$.
- **(b)** Prove $\sum_{n=1}^\infty \frac{1}{n(n+1)} = 1$. *Hint*: Note that $\sum_{k=1}^n \frac{1}{k(k+1)} = \sum_{k=1}^n [\frac{1}{k} - \frac{1}{k+1}]$.
- **(c)** Prove $\sum_{n=1}^\infty \frac{n-1}{2^{n+1}} = \frac{1}{2}$. *Hint*: Note $\frac{k-1}{2^{k+1}} = \frac{k}{2^k} - \frac{k+1}{2^{k+1}}$.
- **(d)** Use (c) to calculate $\sum_{n=1}^\infty \frac{n}{2^n}$.

14.14 Prove $\sum_{n=1}^\infty \frac{1}{n}$ diverges by comparing with the series $\sum_{n=2}^\infty a_n$ where $(a_n)$ is the sequence

$$(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}, \ldots).$$

# §15 Alternating Series and Integral Tests

Sometimes one can check convergence or divergence of series by comparing the partial sums with familiar integrals. We illustrate.

**Example 1**

We show $\sum \frac{1}{n} = +\infty$.

Consider the picture of the function $f(x) = \frac{1}{x}$ in Fig. 15.1. For $n \geq 1$ it is evident that

$$\sum_{k=1}^n \frac{1}{k} = \text{Sum of the areas of the first } n \text{ rectangles in Fig. 15.1}$$

To check the last claim, note that $s_{2k} \le s \le s_{2k+1}$, so both $s_{2k+1}-s$ and $s-s_{2k}$ are clearly bounded by $s_{2k+1}-s_{2k} = a_{2k+1} \le a_{2k}$. So, whether $n$ is even or odd, we have $|s - s_n| \le a_n$. ■

## Exercises

15.1 Determine which of the following series converge. Justify your answers.
**(a)** $\sum \frac{(-1)^n}{n}$ **(b)** $\sum \frac{(-1)^n n!}{2^n}$

15.2 Repeat Exercise 15.1 for the following.
**(a)** $\sum [\sin(\frac{n\pi}{6})]^n$ **(b)** $\sum [\sin(\frac{n\pi}{7})]^n$

15.3 Show $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.

15.4 Determine which of the following series converge. Justify your answers.
**(a)** $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$ **(b)** $\sum_{n=2}^{\infty} \frac{\log n}{n}$
**(c)** $\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log\log n)}$ **(d)** $\sum_{n=2}^{\infty} \frac{\log n}{n^2}$

15.5 Why didn't we use the Comparison Test to prove Theorem 15.1 for $p > 1$?

15.6 **(a)** Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

**(b)** Observe that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges. See Exercise 14.7.

**(c)** Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

15.7 **(a)** Prove if $(a_n)$ is a decreasing sequence of real numbers and if $\sum a_n$ converges, then $\lim n a_n = 0$. *Hint*: Consider $|a_{N+1} + a_{N+2} + \cdots + a_n|$ for suitable $N$.

**(b)** Use (a) to give another proof that $\sum \frac{1}{n}$ diverges.

15.8 Formulate and prove a general integral test as advised in 15.2.

# §16 * Decimal Expansions of Real Numbers

We begin by recalling the brief discussion of decimals in Discussion 10.3. There we considered a decimal expansion $K.d_1 d_2 d_3 \cdots$, As noted in Exercise 9.15, the right-hand side converges to 0. So, for large $n$, the integer $b^n I_n$ lies in the interval $(0, 1)$, a contradiction.

This simplification of Ivan Niven's famous short proof (1947) is due to Zhou and Markov [72]. Zhou and Markov use a similar technique to prove $\tan r$ is irrational for nonzero rational $r$ and $\cos r$ is irrational if $r^2$ is a nonzero rational. Compare with results in Niven's book [49, Chap. 2].

**(c)** It is even more difficult to prove $\pi$ and $e$ are not algebraic numbers; see Definition 2.1. These results are proved in Niven's book [49, Theorems 2.12 and 9.11]. □

**Example 7**
There is a famous number introduced by Euler over 200 years ago that arises in the study of the gamma function. It is known as *Euler's constant* and is defined by

$$\gamma = \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \log_e n\right].$$

Even though

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k} = +\infty \quad \text{and} \quad \lim_{n\to\infty}\log_e n = +\infty,$$

the limit defining $\gamma$ exists and is finite [Exercise 16.9]. In fact, $\gamma$ is approximately 0.577216. The amazing fact is that no one knows whether $\gamma$ is rational or not. Most mathematicians believe $\gamma$ is irrational. This is because it is "easier" for a number to be irrational, since repeating decimal expansions are regular. The remark in Exercise 16.8 hints at another reason it is easier for a number to be irrational. □

## Exercises

16.1 **(a)** Show $2.74\overline{9}$ and $2.75\overline{0}$ are both decimal expansions for $\frac{11}{4}$.

**(b)** Which of these expansions arises from the long division process described in 16.1?

16.2 Verify the claims in the first paragraph of the proof of Theorem 16.3.

16.3 Suppose $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers. Show that if $a_n \le b_n$ for all $n$ and if $a_n < b_n$ for at least one $n$, then $\sum a_n < \sum b_n$.

16.4 Write the following repeating decimals as rationals, i.e., as fractions of integers.

**(a)** $.\overline{2}$ **(b)** $.0\overline{2}$
**(c)** $.\overline{02}$ **(d)** $3.\overline{14}$
**(e)** $.\overline{10}$ **(f)** $.1\overline{492}$

16.5 Find the decimal expansions of the following rational numbers.

**(a)** $1/8$ **(b)** $1/16$
**(c)** $2/3$ **(d)** $7/9$
**(e)** $6/11$ **(f)** $22/7$

16.6 Find the decimal expansions of $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, $\frac{5}{7}$ and $\frac{6}{7}$. Note the interesting pattern.

16.7 Is $.12345678910111213141516171819202122232425 26\cdots$ rational?

16.8 Let $(s_n)$ be a sequence of numbers in $(0,1)$. Each $s_n$ has a decimal expansion $0.d_1^{(n)}d_2^{(n)}d_3^{(n)}\cdots$. For each $n$, let $e_n = 6$ if $d_n^{(n)} \ne 6$ and $e_n = 7$ if $d_n^{(n)} = 6$. Show $.e_1e_2e_3\cdots$ is the decimal expansion for some number $y$ in $(0,1)$ and $y \ne s_n$ for all $n$. *Remark*: This shows the elements of $(0,1)$ cannot be listed as a sequence. In set-theoretic parlance, $(0,1)$ is "uncountable." Since the set $\mathbb{Q} \cap (0,1)$ can be listed as a sequence, there are a lot of irrational numbers in $(0,1)$!

16.9 Let $\gamma_n = (\sum_{k=1}^{n} \frac{1}{k}) - \log_e n = \sum_{k=1}^{n} \frac{1}{k} - \int_1^n \frac{1}{t}dt$.

**(a)** Show $(\gamma_n)$ is a decreasing sequence. *Hint*: Look at $\gamma_n - \gamma_{n+1}$.

**(b)** Show $0 < \gamma_n \le 1$ for all $n$.

**(c)** Observe that $\gamma = \lim_n \gamma_n$ exists and is finite.

16.10 In Example 6(b), we showed $\pi^2$ is irrational. Use this to show $\pi$ is irrational. What can you say about $\sqrt{\pi}$ and $\sqrt[3]{\pi}$? $\pi^4$?