Problem Sheet 2

Use the properties of an ordered field to prove the results in #1 - #6.

1) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

2) If $0 \leq a \leq b$ and $0 \leq c \leq d$, then $ac \leq bd$.

3) If $x > 0$, $y > 0$, and $x^2 < y^2$, then $x < y$.

4) If $0 < x < y$, then $x^n < y^n$ for all $n \in \mathbb{N}$.

5) a) If $0 < c < 1$, then $c^n < c$ for any integer $n > 1$.

   b) If $c > 1$, then $c^n > c$ for any integer $n > 1$.

6) If $a^2 + b^2 = 0$, then $a = 0$ and $b = 0$.

7) Let $a, b \in \mathbb{R}$ with $a \geq 0$, $b \geq 0$. Prove that $\sqrt{ab} \leq \dfrac{a+b}{2}$ using the properties of $\mathbb{R}$ with which you are familiar (without referring to the axioms for an ordered field).

8) Prove that if $r \in \mathbb{Q}$, then there is an $n \in \mathbb{N}$ with $r < n$.