Problem Sheet 3

1) Prove that $\mathbb{Q}$ does not satisfy the Completeness Axiom using the following steps:

Let $E = \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 < 2\}$, and show that E is bounded above in $\mathbb{Q}$.

Next show that sup E does not exist in $\mathbb{Q}$ by giving a proof by contradiction:

A) Assume that $s = \sup E$ exists in $\mathbb{Q}$. Explain how we know that $s^2 \neq 2$.

Now let $r = \dfrac{2s+2}{s+2}$.

B) If $s^2 > 2$, show that $r^2 > 2$ and $0 < r < s$. Explain why this implies that $s \neq \sup E$.

C) If $s^2 < 2$, show that $r^2 < 2$ and $s < r$. Explain why this implies that $s \neq \sup E$.

We can conclude that E is a nonempty subset of $\mathbb{Q}$ which is bounded above, but which has no least upper bound in $\mathbb{Q}$; so $\mathbb{Q}$ does not satisfy the Completeness Axiom.

2) Use the following steps to show that there is a positive real number t such that $t^2 = 3$:

Let $S = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 3\}$.

A) Show that S is nonempty and bounded above, and then let $t = \sup S$.

B) If $t^2 < 3$, use the Archimedean Property to show that there is an $n \in \mathbb{N}$ such that $(t+\frac{1}{n})^2 < 3$; and explain why this gives a contradiction.

C) If $t^2 > 3$, use the Archimedean Property to show that there is an $m \in \mathbb{N}$ such that $(t-\frac{1}{m})^2 > 3$; and explain why this gives a contradiction.

By parts B) and C), we can conclude that $t^2 = 3$.

3) Use the Nested Intervals Property to show that $\mathbb{R}$ is uncountable as follows:

Assume instead that $\mathbb{R}$ is countable, and Let $f : \mathbb{N} \to \mathbb{R}$ be a bijection; so $\mathbb{R} = \{x_1, x_2, x_3, \cdots x_n \cdots\}$ where $x_n = f(n)$ for each $n \in \mathbb{N}$.

Now construct a sequence of nested intervals $I_1, I_2, I_3, \cdots$ with the property that $x_n \notin I_n$ for each $n \in \mathbb{N}$, and use the Nested Intervals Property to get a contradiction.

4) Let F be the field consisting of all quotients $\frac{P(x)}{Q(x)}$ of two polynomials (with real numbers as coefficients), with the usual operations of addition and multiplication. We can make F an ordered field by defining $f > g$ iff $f - g = \frac{P(x)}{Q(x)}$ satisfies $\frac{a}{b} > 0$ where a and b are the leading coefficients of $P(x)$ and $Q(x)$, respectively.

Show that F does not satisfy the Archimedean Property by finding an element in F that is larger than every element of $\mathbb{N}$.

5) Let $S = \{r\sqrt{5} : r \in \mathbb{Q}\}$.

A) Show that S is dense in $\mathbb{R}$.

B) Use part A) to show that the irrational numbers are dense in $\mathbb{R}$.

6) Prove the uniqueness part of the Floor Function Theorem by showing that if $x \in \mathbb{R}$ and $m, n \in \mathbb{Z}$ with $m \leq x < m + 1$ and $n \leq x < n + 1$, then $m = n$.