Problem Sheet 6

1) Let $s_n = \left(1+\frac{1}{n}\right)^n$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} s_n = e$ as follows:

You saw in discussion class that (s_n) converges, so let $\lim_{n\to\infty} s_n = s$.
If $t_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ for all $n \in \mathbb{N}$, we showed in class that (t_n) converges, and

we defined $\lim_{n\to\infty} t_n = e$.

a) Use problem 1b on Discussion Sheet 4 to show that $s \le e$.

b) If $1 \le m \le n$, use the Binomial Theorem to show that

$$s_n \ge 1 + \frac{1}{1!} + \frac{1-\frac{1}{n}}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \cdots + \frac{(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n})}{m!}.$$

c) Use part b) to show that for any $m \in \mathbb{N}$, $s \ge t_m$.

d) Use part c) to show that $s \ge e$.

From parts a) and d), we can conclude that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

2) Let $s_n = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ for all $n \in \mathbb{N}$.

Use the Monotone Convergence Theorem and the fact that

$\frac{1}{m^2} \le \frac{1}{m(m-1)} = \frac{1}{m-1} - \frac{1}{m}$ for $m \ge 2$ to show that (s_n) converges.

3) Let $s_n = \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \cdots + (k(n))\frac{1}{n!}$, where $k(n) = \frac{1}{3}(4\cos((2n-3)\frac{\pi}{3}) + 1)$.

a) Show that $|s_{n+1} - s_n| \le \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

b) Use a problem in Sec. 10 to conclude that (s_n) is a Cauchy sequence.