1) Show that if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ca_n$ diverges for any $c \neq 0$.

2) Show that if $\sum_{n=1}^{\infty} a_n$ diverges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty}(a_n - b_n)$ diverges.

3) Let $\sum_{n=1}^{\infty} a_n$ be a series, and let $\sum_{n=1}^{\infty} b_n$ be a series obtained from $\sum_{n=1}^{\infty} a_n$ by grouping (inserting parentheses around groups of finitely many terms of $\sum_{n=1}^{\infty} a_n$).

Show that if $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges and has the same sum.

4) Let $\sum_{n=1}^{\infty} a_n$ be a series and let m be in $\mathbb{N}$.

Show that $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=m+1}^{\infty} a_n$ converges, and that $\sum_{n=1}^{\infty} a_n = S_m + \sum_{n=m+1}^{\infty} a_n$ when both series converge.

5) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with $a_n = b_n$ for $n \geq N$, for some N in $\mathbb{N}$.

Prove that $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} b_n$ converges.

6) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series.

If $a_n \leq b_n$ for all n in $\mathbb{N}$, and $a_m < b_m$ for some m in $\mathbb{N}$, show that $\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} b_n$.

7) Use the following steps to show that the harmonic series diverges:

a) Show that $S_{2^m} \geq 1 + \frac{m}{2}$ for all m in $\mathbb{N}$ using induction.

b) Use part a) to show that $\lim_{n\to\infty} S_n = \infty$.

8) Identify the results on this sheet which are used in the following proof that the harmonic series diverges:

Assume instead that the harmonic series converges, with $\sum_{n=1}^{\infty} \frac{1}{n} = S$.

Then $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots > \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{6} + \frac{1}{8} + \frac{1}{8} + \cdots$

$= \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{6} + \frac{1}{6}\right) + \left(\frac{1}{8} + \frac{1}{8}\right) + \cdots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = S$, which gives a contradiction.

9) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with all terms nonnegative.

Prove that if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} a_n b_n$ converges.