Math 25 Problem Sheet 8

1) Prove or disprove the following statements:

a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are positive-term series and $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

b) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

2) Prove that absolute convergence implies convergence using the following steps:

a) Show that $0 \le x + |x| \le 2|x|$ for every number x.

b) Use part a) to show that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges.

c) Use part b) to show that if $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

3) Let (S_n) be the sequence of partial sums for the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$, and let (H_n) be the sequence of partial sums for the harmonic series.

a) Show that $S_{2n} = H_{2n} - H_n$ for all n in $\mathbb{N}$.

b) Find $\lim_{n\to\infty} S_{2n}$ using part a) and a problem from Discussion sheet 4.

c) Show that $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} S_{2n+1}$.

d) Find the sum of the alternating harmonic series using parts b) and c).

4) Explain why the Alternating Series Test does not apply to the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\frac{1}{2}(3+(-1)^{n+1})}{\frac{1}{4}(2n+1+(-1)^{n+1})}$, and then determine if it converges or diverges by considering its sequence of partial sums.

5) Let $\sum_{n=1}^{\infty} a_n$ be a series whose terms are positive and decreasing, and

let $\sum_{n=1}^{\infty} 2^n a_{2^n} = 2a_2 + 4a_4 + 8a_8 + 16a_{16} + \cdots + 2^n a_{2^n} + \cdots$.

Show that $\sum_{n=1}^{\infty} a_n$ converges iff $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges using the Comparison Test with

i) $\sum_{n=2}^{\infty} a_n$ and $\frac{1}{2}\sum_{n=1}^{\infty} 2^n a_{2^n} = a_2 + a_4 + a_4 + a_8 + a_8 + a_8 + a_8 + a_{16} + \cdots$

ii) $\sum_{n=2}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 2^n a_{2^n} = a_2 + a_2 + a_4 + a_4 + a_4 + a_4 + a_8 + \cdots$.

6) Use the result of problem 5 [the **Cauchy Condensation Test**] to determine if the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

7) Define (s_n) by $s_n = (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}) - \ln n$ for all n in $\mathbb{N}$.

a) Show that (s_n) is decreasing.

b) Use part a) to show that (s_n) converges. (Its limit is called Euler's constant, and is denoted by γ.)

c) Use part b), together with part a) of problem 3, to find the sum of the alternating harmonic series.

8) Find the sum of the rearrangement $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$ of the alternating harmonic series using problem 3 and the following steps:

a) Multiply each term of the alternating harmonic series by $\frac{1}{2}$.

b) Insert a zero term between each term of this series (and at the start of the series).

c) Add this series to the alternating harmonic series term-by-term, and omit zero terms.

9) **Bonus question**

If $\sigma(n)$ is the sum of the divisors of the integer n and H_n is the nth partial sum of the harmonic series, show that $\sigma(n) < H_n + e^{H_n} \ln(H_n)$ for all integers $n > 1$.