

Math 17B

Vogler

Using Eigenvalues and Eigenvectors to Solve Matrix Algebra Problems

DERIVE A FORMULA FOR POWERS OF A MATRIX APPLIED TO ITS EIGENVECTOR :

Let A be a matrix and assume that λ is an eigenvalue with eigenvector V . Then

$$AV = \lambda V \quad \longrightarrow$$

$$A^2V = A(AV) = A(\lambda V) = \lambda(AV) = \lambda(\lambda V) = \lambda^2V \quad \longrightarrow$$

$$A^3V = A(A^2V) = A(\lambda^2V) = \lambda^2(AV) = \lambda^2(\lambda V) = \lambda^3V \quad \longrightarrow$$

$$A^4V = A(A^3V) = A(\lambda^3V) = \lambda^3(AV) = \lambda^3(\lambda V) = \lambda^4V \quad \longrightarrow$$

$$(P) \quad A^kV = \lambda^kV \quad \text{for } k = 1, 2, 3, 4, \dots$$

USE THE ABOVE FORMULA TO SOLVE THE FOLLOWING PROBLEM :

EXAMPLE : Consider matrix $A = \begin{pmatrix} -1/2 & 5/4 \\ 1 & 3/2 \end{pmatrix}$. It can be shown that it's eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$ with eigenvectors $V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $V_2 = \begin{pmatrix} -5 \\ 2 \end{pmatrix}$, resp.

Use the eigenvalues and eigenvectors for matrix A to determine $A^{30} \begin{pmatrix} 13 \\ 2 \end{pmatrix}$.

Begin by writing vector $\begin{pmatrix} 13 \\ 2 \end{pmatrix}$ as a linear combination of the eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$:

$$\begin{pmatrix} 13 \\ 2 \end{pmatrix} = (c_1) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c_2) \begin{pmatrix} -5 \\ 2 \end{pmatrix} \quad \longrightarrow$$

$$13 = c_1 - 5c_2$$

$$2 = 2c_1 + 2c_2 \quad \longrightarrow$$

$$-26 = -2c_1 + 10c_2$$

$$2 = 2c_1 + 2c_2 \quad \longrightarrow \quad -24 = 12c_2 \quad \longrightarrow \quad c_2 = -2 \quad \text{and} \quad c_1 = 3 .$$

Thus, we have $\begin{pmatrix} 13 \\ 2 \end{pmatrix} = (3) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-2) \begin{pmatrix} -5 \\ 2 \end{pmatrix} \quad \longrightarrow$

$$\begin{aligned}
A^{30} \binom{13}{2} &= A^{30} \left((3) \binom{1}{2} + (-2) \binom{-5}{2} \right) \\
&= A^{30} \left((3) \binom{1}{2} \right) + A^{30} \left((-2) \binom{-5}{2} \right) \\
&= (3) \left(A^{30} \binom{1}{2} \right) + (-2) \left(A^{30} \binom{-5}{2} \right)
\end{aligned}$$

(Now use equation (P).)

$$\begin{aligned}
&= (3) \left((2)^{30} \binom{1}{2} \right) + (-2) \left((-1)^{30} \binom{-5}{2} \right) \\
&= \binom{(3)(2)^{30}}{(3)(2)^{31}} + \binom{10}{-4} \\
&= \binom{3, 221, 225, 482}{6, 442, 450, 940}.
\end{aligned}$$