QUESTION : Consider the function \( y = f(x) \) and assume that mathematical circumstances would require that this function be replaced with an \( n \)th-degree polynomial whose values closely approximate the values of \( y = f(x) \). How would we determine the coefficients of this unknown polynomial?

ANSWER : Assume that \( y = f(x) \) is a given function and constant "a" is known. Determine a list of real numbers \( a_0, a_1, a_2, a_3, \ldots, a_n \) so that the polynomial is

\[
P_n(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \cdots + a_n(x - a)^n,
\]

which satisfies the condition

\[
f(x) \approx P_n(x),
\]

i.e.,

\[(T) \quad f(x) \approx a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \cdots + a_n(x - a)^n.
\]

If we assume that "\( \approx \)" is "\( = \)" and substitute \( x = a \) in equation \((T)\), we get

\[
f(a) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \cdots + a_n(0)^n = a_0,
\]

i.e.,

\[
a_0 = f(a).
\]

Now differentiate equation \((T)\) term by term getting

\[
f'(x) = a_1 + 2a_2(x - a) + 3a_3(x - a)^2 + 4a_4(x - a)^3 + \cdots + na_n(x - a)^{n-1}.
\]

If we substitute \( x = a \) in this equation, we get

\[
f'(a) = a_1 + 2a_2(0) + 3a_3(0)^2 + 4a_4(0)^3 + \cdots + na_n(0)^{n-1} = a_1,
\]

i.e.,

\[
a_1 = f'(a).
\]

Now differentiate again term by term getting

\[
f''(x) = 2a_2 + 3 \cdot 2a_3(x - a) + 4 \cdot 3a_4(x - a)^2 + 5 \cdot 4a_5(x - a)^3 + \cdots + n \cdot (n - 1)a_n(x - a)^{n-2}.
\]

If we substitute \( x = a \) in this equation, we get

\[
f''(a) = 2a_2 + 3 \cdot 2a_3(0) + 4 \cdot 3a_4(0)^2 + 5 \cdot 4a_5(0)^3 + \cdots + (n - 1) \cdot na_n(0)^{n-2} = 2a_2,
\]

i.e.,

\[
a_2 = \frac{f''(a)}{2!}.
\]

Now differentiate again term by term getting

\[
f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(x - a) + 5 \cdot 4 \cdot 3a_5(x - a)^2 + \cdots + n(n - 1)(n - 2)a_n(x - a)^{n-3}.
\]
If we substitute $x = a$ in this equation, we get
\[ f'''(a) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4(0) + 5 \cdot 4 \cdot 3a_5(0)^2 + \cdots + n(n-1)(n-2)a_n(0)^{n-3} = 3 \cdot 2a_3, \]
i.e.,
\[ a_3 = \frac{f'''(a)}{3!}. \]
Continuing this term by term differentiation and substitution process results in the fact that
\[ a_k = \frac{f^{(k)}(a)}{k!} \quad \text{for} \quad k = 0, 1, 2, 3, \ldots, n. \]

**DEFINITION**: The *Taylor Polynomial of degree n* centered at $x = a$ for function $y = f(x)$ is given by
\[ P_n(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + a_3(x - a)^3 + \cdots + a_n(x - a)^n \]
and
\[ a_k = \frac{f^{(k)}(a)}{k!} \quad \text{for} \quad k = 0, 1, 2, 3, \ldots, n. \]

**QUESTION**: Consider an ordinary function $y = f(x)$ and $P_n(x)$, its Taylor Polynomial of degree $n$ centered at $x = a$. We assume that $f(x) \approx P_n(x)$. Let the error be given by
\[ R_{n+1}(x; a) = f(x) - P_n(x). \] It can be shown that the absolute error (Taylor Error) for their difference is
\[ \left| R_{n+1}(x; a) \right| = \left| \frac{f^{(n+1)}(c)}{(n + 1)!} (x - a)^{n+1} \right|, \]
where $c$ is some number between $x$ and $a$. (Note that the number $c$ appears as a result of using the Intermediate Value Theorem to derive this error formula.)