Now, for $a > 1$,

$$\int_1^a f(x)\,dx < 0$$

Therefore,

$$\int_0^a f(x)\,dx = \int_0^1 f(x)\,dx + \underbrace{\int_1^a f(x)\,dx}_{<0} < \int_0^1 f(x)\,dx \tag{6.3}$$

Combining (6.2) and (6.3) shows that

$$\int_0^a f(x)\,dx < \int_0^1 f(x)\,dx$$

for all $a \geq 0$ and $a \neq 1$. Hence, $a = 1$ maximizes the integral $\int_0^a (1 - x^2)\,dx$. ■

Section 6.1 Problems

■ 6.1.1

1. Approximate the area under the parabola $y = x^2$ from 0 to 1, using four equal subintervals with left endpoints.

2. Approximate the area under the parabola $y = x^2$ from 0 to 1, using five equal subintervals with midpoints.

3. Approximate the area under the parabola $y = x^2$ from 0 to 1, using four equal subintervals with right endpoints.

4. Approximate the area under the parabola $y = 1 - x^2$ from 0 to 1, using five equal subintervals with **(a)** left endpoints and **(b)** right endpoints.

In Problems 5–14, write each sum in expanded form.

5. $\sum_{k=1}^{4} \sqrt{k}$

6. $\sum_{k=3}^{5} (k-1)^2$

7. $\sum_{k=2}^{6} 3^k$

8. $\sum_{k=1}^{3} \frac{k^2}{k^2+1}$

9. $\sum_{k=0}^{3} (x+1)^k$

10. $\sum_{k=0}^{4} k^x$

11. $\sum_{k=0}^{3} (-1)^{k+1}$

12. $\sum_{k=1}^{n} f(c_k)\Delta x_k$

13. $\sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \frac{1}{n}$

14. $\sum_{k=1}^{n} \cos\left(k\frac{\pi}{n}\right)\frac{\pi}{n}$

In Problems 15–22, write each sum in sigma notation.

15. $2 + 4 + 6 + 8 + \cdots + 2n$

16. $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}}$

17. $\ln 2 + \ln 3 + \ln 4 + \ln 5$

18. $\frac{3}{5} + \frac{4}{6} + \frac{5}{7} + \frac{6}{8} + \frac{7}{9}$

19. $-\frac{1}{4} + \frac{1}{6} + \frac{2}{7} + \frac{3}{8}$

20. $\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n}$

21. $1 + q + q^2 + q^3 + q^4 + \cdots + q^{n-1}$

22. $1 - a + a^2 - a^3 + a^4 - a^5 + \cdots + (-1)^n a^n$

In Problems 23–30, use the algebraic rules for sums to evaluate each sum. Recall that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

and

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

23. $\sum_{k=1}^{15} (2k+3)$

24. $\sum_{k=1}^{5} (4-k^2)$

25. $\sum_{k=0}^{6} k(k+1)$

26. $\sum_{k=1}^{n} 4k$

27. $\sum_{k=1}^{n} 4(k-1)^2$

28. $\sum_{k=1}^{n} (k+2)(k-2)$

29. $\sum_{k=1}^{10} (-1)^k$

30. $\sum_{k=0}^{10} (-1)^k$

31. The steps that follow will show that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) Show that

$$\begin{aligned}\sum_{k=1}^{n} [(1+k)^3 - k^3] &= (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) \\ &\quad + \cdots + [(1+n)^3 - n^3] \\ &= (1+n)^3 - 1^3\end{aligned}$$

(Sums that "collapse" like this due to cancellation of terms are called *telescoping* or *collapsing* sums.)

(b) Use Example 3 and the algebraic rules for sums to show that

$$\sum_{k=1}^{n} [(1+k)^3 - k^3] = 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n$$

(c) In (a) and (b), we found two expressions for the sum

$$\sum_{k=1}^{n}[(1+k)^3 - k^3]$$

Those two expressions are therefore equal; that is,

$$(1+n)^3 - 1^3 = 3\sum_{k=1}^{n} k^2 + 3\frac{n(n+1)}{2} + n$$

Solve this equation for $\sum_{k=1}^{n} k^2$, and show that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

■ 6.1.2

32. Approximate

$$\int_{-1}^{1}(1-x^2)\,dx$$

using five equal subintervals and left endpoints.

33. Approximate

$$\int_{-1}^{1}(1-x^2)\,dx$$

using five equal subintervals and midpoints.

34. Approximate

$$\int_{-1}^{1}(2+x^2)\,dx$$

using five equal subintervals and right endpoints.

35. Approximate

$$\int_{-2}^{2}(2+x^2)\,dx$$

using four equal subintervals and left endpoints.

36. Approximate

$$\int_{-1}^{2} e^{-x}\,dx$$

using three equal subintervals and midpoints.

37. Approximate

$$\int_{0}^{3\pi/2} \sin x\,dx$$

using three equal subintervals and right endpoints.

38. (a) Assume that $a > 0$. Evaluate $\int_0^a x\,dx$, using the fact that the region bounded by $y = x$ and the x-axis between 0 to a is a triangle. (See Figure 6.23.)

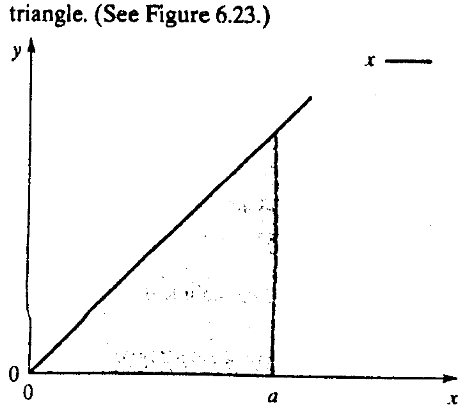

Figure 6.23 The region for Problem 38.

(b) Assume that $a > 0$. Evaluate $\int_0^a x\,dx$ by approximating the region bounded by $y = x$ and the x-axis from 0 to a with rectangles. Use equal subintervals and take right endpoints. (*Hint*: Use the result in Example 3 to evaluate the sum of the areas of the rectangles.)

39. Assume that $0 < a < b < \infty$. Use a geometric argument to show that

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}$$

40. Assume that $0 < a < b < \infty$. Use a geometric argument and Example 1 to show that

$$\int_a^b x^2\,dx = \frac{b^3 - a^3}{3}$$

Express the limits in Problems 41–47 as definite integrals. Note that (1) $P = [x_0, x_1, \ldots, x_n]$ is a partition of the indicated interval, (2) $c_k \in [x_{k-1}, x_k]$, and (3) $\Delta x_k = x_k - x_{k-1}$.

41. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} 2c_k^3 \Delta x_k$, where P is a partition of $[1, 2]$

42. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \sqrt{c_k}\,\Delta x_k$, where P is a partition of $[1, 4]$

43. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (2c_k - 1)\Delta x_k$, where P is a partition of $[-3, 2]$

44. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{1}{c_k + 1}\Delta x_k$, where P is a partition of $[1, 2]$

45. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} \frac{c_k - 1}{c_k + 2}\Delta x_k$, where P is a partition of $[2, 3]$

46. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} (\sin c_k)\Delta x_k$, where P is a partition of $[0, \pi]$

47. $\lim_{\|P\|\to 0} \sum_{k=1}^{n} e^{c_k}\Delta x_k$, where P is a partition of $[-5, 2]$

In Problems 48–53, express the definite integrals as limits of Riemann sums.

48. $\int_{-2}^{-1} \frac{x^2}{1+x^2}\,dx$ **49.** $\int_{2}^{6} (x+1)^{1/3}\,dx$

50. $\int_{1}^{3} e^{-2x}\,dx$ **51.** $\int_{1}^{e} \ln x\,dx$ **52.** $\int_{0}^{\pi} \cos\frac{2x}{\pi}\,dx$

53. $\int_{0}^{5} g(x)\,dx$, where $g(x)$ is a continuous function on $[0, 5]$

In Problems 54–60, use a graph to interpret the definite integral in terms of areas. Do not compute the integrals.

54. $\int_{0}^{3} (2x+1)\,dx$ **55.** $\int_{-1}^{2} (x^2 - 1)\,dx$

56. $\int_{-2}^{2} \frac{1}{2}x^3\,dx$ **57.** $\int_{0}^{5} e^{-x}\,dx$

58. $\int_{-\pi}^{\pi} \cos x\,dx$ **59.** $\int_{1/2}^{4} \ln x\,dx$

60. $\int_{-3}^{2} \left(1 - \frac{1}{2}x\right)\,dx$

In Problems 61–67, use an area formula from geometry to find the value of each integral by interpreting it as the (signed) area under the graph of an appropriately chosen function.

61. $\int_{-2}^{3} |x|\,dx$

62. $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

63. $\int_{2}^{5} \left(\frac{1}{2}x-4\right) dx$

64. $\int_{1/2}^{1} \sqrt{1-x^2}\,dx$

65. $\int_{-2}^{2} (\sqrt{4-x^2}-2)\,dx$

66. $\int_{0}^{1} \sqrt{2-x^2}\,dx$

67. $\int_{-3}^{0} (4-\sqrt{9-x^2})\,dx$

■ 6.1.3

68. Given that

$$\int_0^a x^2\,dx = \frac{1}{3}a^3$$

evaluate the following:

(a) $\int_0^2 \frac{1}{2}x^2\,dx$ **(b)** $\int_{-3}^{-2} 3x^2\,dx$

(c) $\int_{-1}^{3} \frac{1}{3}x^2\,dx$ **(d)** $\int_{1}^{1} 3x^2\,dx$

(e) $\int_{-2}^{3} (x+1)^2\,dx$ **(f)** $\int_{2}^{4} (x-2)^2\,dx$

69. Find $\int_2^2 \cos(3x^2)\,dx$.

70. Find $\int_{-3}^{-3} e^{-x^2/2}\,dx$.

71. Find $\int_{-2}^{2} \frac{x^3}{3}\,dx$.

72. Find $\int_{-5}^{5} 2x^5\,dx$.

73. Find $\int_{-1}^{1} \tan x\,dx$.

74. Explain geometrically why

$$\int_1^2 x^2\,dx = \int_0^2 x^2\,dx - \int_0^1 x^2\,dx \tag{6.4}$$

and show that (6.4) can be written as

$$\int_1^2 x^2\,dx = \int_1^0 x^2\,dx + \int_0^2 x^2\,dx \tag{6.5}$$

Relate (6.5) to addition property (5).

In Problems 75–79, verify each inequality without evaluating the integrals.

75. $\int_0^1 x\,dx \geq \int_0^1 x^2\,dx$

76. $\int_1^2 x\,dx \leq \int_1^2 x^2\,dx$

77. $0 \leq \int_0^4 \sqrt{x}\,dx \leq 8$

78. $\frac{1}{2} \leq \int_0^1 \sqrt{1-x^2}\,dx \leq 1$

79. $\frac{\pi}{3} \leq \int_{\pi/6}^{5\pi/6} \sin x\,dx \leq \frac{2\pi}{3}$

80. Find the value of $a \geq 0$ that maximizes $\int_0^a (4-x^2)\,dx$.

81. Find the value of $a \in [0, 2\pi]$ that maximizes $\int_0^a \cos x\,dx$.

82. Find $a \in (0, 2\pi]$ such that

$$\int_0^a \sin x\,dx = 0$$

83. Find $a > 1$ such that

$$\int_1^a (x-2)^3\,dx = 0$$

84. Find $a > 0$ such that

$$\int_{-a}^a (1-|x|)\,dx = 0$$

85. To determine age-specific mortality, a group of individuals, all born at the same time, is followed over time. If $N(t)$ denotes the number still alive at time t, then $N(t)/N(0)$ is the fraction surviving at time t. The quantity $r(t)$, called the *hazard rate function*, measures the rate at which individuals die at time t; that is, $r(t)\,dt$ is the probability that an individual who is alive at time t dies during the infinitesimal time interval $(t, t+dt)$. The cumulative hazard during the time interval $[0, t]$, $\int_0^t r(s)\,ds$, can be estimated as $-\ln \frac{N(t)}{N(0)}$. Show that the cumulative hazard during the time interval $[t, t+1]$, $\int_t^{t+1} r(s)\,ds$, can be estimated as $-\ln \frac{N(t+1)}{N(t)}$.

■ 6.2 The Fundamental Theorem of Calculus

In Section 6.1, we used the definition of definite integrals to compute $\int_0^a x^2\,dx$. This required the summation of a large number of terms, which was facilitated by the explicit summation formula for $\sum_{k=1}^n k^2$. Fermat and others were able to carry out similar calculations for the area under curves of the form $y = x^r$, where r was a rational number different from -1. The solution to the case $r = -1$ was found by the Belgian mathematician Gregory of St. Vincent (1584–1667) and published in 1647. At that time, it seemed that methods specific to a given function needed to be developed to compute the area under the curve of that function. Such methods would not have been practical.

Fortunately, it turns out that the area problem is related to the tangent problem. This relationship is not at all obvious; among the first to notice it were Isaac Barrow (1630–1677) and James Gregory (1638–1675). Each presented the relationship in geometrical terms, without realizing the importance of his discovery.

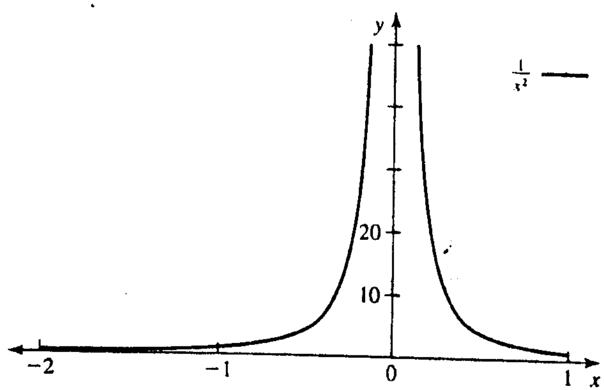

Figure 6.26 The graph of $y = \frac{1}{x^2}$ between $x = -2$ and $x = 1$. The function is discontinuous at $x = 0$.

Section 6.2 Problems

■ 6.2.1

In Problems 1–14, find $\frac{dy}{dx}$.

1. $y = \int_0^x 2u^3 du$
2. $y = \int_0^x (1 - \frac{u^4}{2}) du$
3. $y = \int_0^x (4u^2 - 3) du$
4. $y = \int_0^x (3 + u^4) du$
5. $y = \int_0^x \sqrt{1 + 2u}\, du,\ x > 0$
6. $y = \int_0^x \sqrt{1 + u^2}\, du,\ x > 0$
7. $y = \int_0^x \sqrt{1 + \sin^2 u}\, du,\ x > 0$
8. $y = \int_0^x \sqrt{2 + \csc^2 u}\, du,\ x > 0$
9. $y = \int_3^x u e^{4u} du$
10. $y = \int_1^x u e^{-u^2} du$
11. $y = \int_{-2}^x \frac{1}{u + 3} du,\ x > -2$
12. $y = \int_{-1}^x \frac{2}{2 + u^2} du$
13. $y = \int_{\pi/2}^x \sin(u^2 + 1)\, du$
14. $y = \int_{\pi/4}^x \cos^2(u - 3)\, du$

In Problems 15–38, use Leibniz's rule to find $\frac{dy}{dx}$.

15. $y = \int_0^{3x} (1 + t^2)\, dt$
16. $y = \int_0^{2x-1} (t^3 - 2)\, dt$
17. $y = \int_0^{1-4x} (2t^2 + 1)\, dt$
18. $y = \int_0^{3x+2} (1 + t^3)\, dt$
19. $y = \int_4^{x^2+1} \sqrt{t}\, dt,\ x > 0$
20. $y = \int_2^{x^2-2} \sqrt{3 + u}\, du,\ x > 0$
21. $y = \int_0^{3x} (1 + e^t)\, dt$
22. $y = \int_0^{2x^2-1} (e^{-2t} + e^{2t})\, dt$
23. $y = \int_1^{3x^2+x} (1 + te^t)\, dt$
24. $y = \int_2^{\ln x} e^{-t}\, dt,\ x > 0$
25. $y = \int_x^3 (1 + t)\, dt$
26. $y = \int_x^5 (1 + e^t)\, dt$
27. $y = \int_{2x}^3 (1 + \sin t)\, dt$
28. $y = \int_{2x^2}^6 (1 + \tan t)\, dt$
29. $y = \int_x^5 \frac{1}{u^2} du,\ x > 0$
30. $y = \int_{x^2}^3 \frac{1}{1 + t} dt,\ x > 0$
31. $y = \int_{x^2}^1 \sec t\, dt,\ -1 < x < 1$
32. $y = \int_{2+x^2}^2 \cot t\, dt$
33. $y = \int_x^{2x} (1 + t^2)\, dt$
34. $y = \int_{-x}^x \tan u\, du,\ 0 < x < \frac{\pi}{4}$
35. $y = \int_{x^2}^{x^3} \ln(t - 3)\, dt,\ x > 0$
36. $y = \int_{x^3}^{x^4} \ln(1 + t^2)\, dt,\ x > 0$
37. $y = \int_{2-x^2}^{x+x^3} \sin t\, dt$
38. $y = \int_{1+x^2}^{x^3-2x} \cos t\, dt$

■ 6.2.2

In Problems 39–96, compute the indefinite integrals.

39. $\int (1 + 3x^2)\, dx$
40. $\int (x^3 - 4)\, dx$
41. $\int (\frac{1}{3}x^2 - \frac{1}{2}x)\, dx$
42. $\int (4x^3 + 5x^2)\, dx$
43. $\int \left(\frac{1}{2}x^2 + 3x - \frac{1}{3}\right) dx$
44. $\int \left(\frac{1}{2}x^5 + 2x^3 - 1\right) dx$
45. $\int \frac{2x^2 - x}{\sqrt{x}} dx$
46. $\int \frac{x^3 + 3x}{2\sqrt{x}} dx$
47. $\int x^2 \sqrt{x}\, dx$
48. $\int (1 + x^3)\sqrt{x}\, dx$
49. $\int (x^{7/2} + x^{2/7})\, dx$
50. $\int (x^{3/5} + x^{5/3})\, dx$
51. $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$
52. $\int \left(3x^{1/3} + \frac{1}{3x^{1/3}}\right) dx$
53. $\int (x - 1)(x + 1)\, dx$
54. $\int (x - 1)^2\, dx$

55. $\int (x-2)(3-x)\,dx$

56. $\int (2x+3)^2\,dx$

57. $\int e^{2x}\,dx$

58. $\int 2e^{3x}\,dx$

59. $\int 3e^{-x}\,dx$

60. $\int 2e^{-x/3}\,dx$

61. $\int xe^{-x^2/2}\,dx$

62. $\int e^x(1-e^{-x})\,dx$

63. $\int \sin(2x)\,dx$

64. $\int \sin\frac{1-x}{3}\,dx$

65. $\int \cos(3x)\,dx$

66. $\int \cos\frac{2-4x}{5}\,dx$

67. $\int \sec^2(3x)\,dx$

68. $\int \csc^2(2x)\,dx$

69. $\int \frac{\sin x}{1-\sin^2 x}\,dx$

70. $\int \frac{\cos x}{1-\cos^2 x}\,dx$

71. $\int \tan(2x)\,dx$

72. $\int \cot(3x)\,dx$

73. $\int (\sec^2 x + \tan x)\,dx$

74. $\int (\cot x - \csc^2 x)\,dx$

75. $\int \frac{4}{1+x^2}\,dx$

76. $\int \left(1-\frac{x^2}{1+x^2}\right)\,dx$

77. $\int \frac{1}{\sqrt{1-x^2}}\,dx$

78. $\int \frac{5}{\sqrt{1-x^2}}\,dx$

79. $\int \frac{1}{x+2}\,dx$

80. $\int \frac{1}{x-3}\,dx$

81. $\int \frac{2x-1}{3x}\,dx$

82. $\int \frac{2x+5}{x}\,dx$

83. $\int \frac{x+3}{x^2-9}\,dx$

84. $\int \frac{x+4}{x^2-16}\,dx$

85. $\int \frac{3-x}{x^2-9}\,dx$

86. $\int \frac{4-x}{x^2-16}\,dx$

87. $\int \frac{5x^2}{x^2+1}\,dx$

88. $\int \frac{2x^2}{1+x^2}\,dx$

89. $\int 3^x\,dx$

90. $\int 2^x\,dx$

91. $\int 3^{-2x}\,dx$

92. $\int 4^{-x}\,dx$

93. $\int (x^2+2^x)\,dx$

94. $\int (x^{-3}+3^{-x})\,dx$

95. $\int (\sqrt{x}+\sqrt{e^x})\,dx$

96. $\int \left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{e^x}}\right)\,dx$

■ 6.2.3

In Problems 97–122, evaluate the definite integrals.

97. $\int_2^4 (3-2x)\,dx$

98. $\int_{-1}^3 (2x^2-1)\,dx$

99. $\int_0^1 (x^3-x^{1/3})\,dx$

100. $\int_1^2 x^{5/2}\,dx$

101. $\int_1^8 x^{-2/3}\,dx$

102. $\int_4^9 \frac{1+\sqrt{x}}{\sqrt{x}}\,dx$

103. $\int_0^2 (2t-1)(t+3)\,dt$

104. $\int_{-1}^2 (2+3t)^2\,dt$

105. $\int_0^{\pi/4} \sin(2x)\,dx$

106. $\int_{-\pi/3}^{\pi/3} 2\cos\left(\frac{x}{2}\right)\,dx$

107. $\int_0^{\pi/8} \sec^2(2x)\,dx$

108. $\int_{-\pi/4}^{\pi/4} \tan x\,dx$

109. $\int_0^1 \frac{1}{1+x^2}\,dx$

110. $\int_{-\sqrt{3}}^{-1} \frac{4}{1+x^2}\,dx$

111. $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$

112. $\int_{-1/2}^{1/2} \frac{2}{\sqrt{1-x^2}}\,dx$

113. $\int_0^{\pi/6} \tan(2x)\,dx$

114. $\int_{\pi/20}^{\pi/15} \sec(5x)\tan(5x)\,dx$

115. $\int_{-1}^0 e^{3x}\,dx$

116. $\int_0^2 2te^{t^2}\,dt$

117. $\int_{-1}^1 |x|\,dx$

118. $\int_{-1}^1 e^{-|s|}\,ds$

119. $\int_1^e \frac{1}{x}\,dx$

120. $\int_2^3 \frac{1}{z+1}\,dz$

121. $\int_{-2}^{-1} \frac{1}{1-u}\,du$

122. $\int_2^3 \frac{2}{t-1}\,dt$

123. Use l'Hospital's rule to compute

$$\lim_{x\to 0}\frac{1}{x^2}\int_0^x \sin t\,dt$$

124. Use l'Hospital's rule to compute

$$\lim_{h\to 0}\frac{1}{h}\int_0^h e^x\,dx$$

125. Suppose that

$$\int_0^x f(t)\,dt = 2x^2$$

Find $f(x)$.

126. Suppose that

$$\int_0^x f(t)\,dt = \frac{1}{2}\tan(2x)$$

Find $f(x)$.

■ 6.3 Applications of Integration

In this section, we will discuss a number of applications of integrals. In the first application, we will revisit the interpretation of integrals as areas; the second application interprets integrals as cumulative (or net) change; the third will allow us to compute averages using integrals; and, finally, we will use integrals to compute volumes. In each application, you will see that integrals can be interpreted as "sums of many small increments."

EXAMPLE 13 Set up, but do not evaluate, the length of the curve of the hyperbola $f(x) = \frac{1}{x}$ between $a = 1$ and $b = 2$.

Solution To determine the length of the curve, we need to find $f'(x)$ first.

$$f'(x) = -\frac{1}{x^2}$$

Then the length of the curve is given by the integral

$$L = \int_1^2 \sqrt{1 + \left(-\frac{1}{x^2}\right)^2}\, dx = \int_1^2 \sqrt{1 + \frac{1}{x^4}}\, dx$$ ■

The antiderivative of the integrand in Example 13 is quite complicated, and we will not be able to find it with the techniques available in this text. In Section 7.5, we will learn numerical methods for evaluating integrals, and some of these methods can be used to evaluate the integral in Example 13. There are also computer software packages that can numerically evaluate such integrals. Using either of these approaches on the integral in Example 13, we would find that the length L is approximately 1.13.

Section 6.3 Problems

■ 6.3.1

Find the areas of the regions bounded by the lines and curves in Problems 1–12.

1. $y = x^2 - 4,\ y = x + 2$
2. $y = 2x^2 - 1,\ y = 2 - x^4$
3. $y = e^{x/2},\ y = -x,\ x = 0,\ x = 2$
4. $y = \cos x,\ y = 1,\ x = 0,\ x = \frac{\pi}{2}$
5. $y = x^2 + 1,\ y = 4x - 2$ (in the first quadrant)
6. $y = x^2,\ y = 2 - x,\ y = 0$ (in the first quadrant)
7. $y = x^2,\ y = \frac{1}{x},\ y = 4$ (in the first quadrant)
8. $y = \sin x,\ y = \cos x$ from $x = 0$ to $x = \frac{\pi}{4}$
9. $y = \sin x,\ y = 1$ from $x = 0$ to $x = \frac{\pi}{2}$
10. $y = x^2,\ y = (x-2)^2,\ y = 0$ from $x = 0$ to $x = 2$
11. $y = x^2,\ y = x^3$ from $x = 0$ to $x = 2$
12. $y = e^{-x},\ y = x + 1$ from $x = -1$ to $x = 1$

In Problems 13–16, find the areas of the regions bounded by the lines and curves by expressing x as a function of y and integrating with respect to y.

13. $y = x^2,\ y = (x-2)^2,\ y = 0$ from $x = 0$ to $x = 2$
14. $y = x,\ yx = 1,\ y = \frac{1}{2}$ (in the first quadrant)
15. $x = (y-1)^2 + 3,\ x = 1 - (y-1)^2$ from $y = 0$ to $y = 2$ (in the first quadrant)
16. $x = (y-1)^2 - 1,\ x = (y-1)^2 + 1$ from $y = 0$ to $y = 2$

■ 6.3.2

17. Consider a population whose size at time t is $N(t)$ and whose dynamics are given by the initial-value problem

$$\frac{dN}{dt} = e^{-t}$$

with $N(0) = 100$.

(a) Find $N(t)$ by solving the initial-value problem.

(b) Compute the cumulative change in population size between $t = 0$ and $t = 5$.

(c) Express the cumulative change in population size between time 0 and time t as an integral. Give a geometric interpretation of this quantity.

18. Suppose that a change in biomass $B(t)$ at time t during the interval $[0, 12]$ follows the equation

$$\frac{d}{dt}B(t) = \cos\left(\frac{\pi}{6}t\right)$$

for $0 \le t \le 12$.

(a) Graph $\frac{dB}{dt}$ as a function of t.

(b) Suppose that $B(0) = B_0$. Express the cumulative change in biomass during the interval $[0, t]$ as an integral. Give a geometric interpretation. What is the value of the biomass at the end of the interval $[0, 12]$ compared with the value at time 0? How are these two quantities related to the cumulative change in the biomass during the interval $[0, 12]$?

19. A particle moves along the x-axis with velocity

$$v(t) = -(t-2)^2 + 1$$

for $0 \le t \le 5$. Assume that the particle is at the origin at time 0.

(a) Graph $v(t)$ as a function of t.

(b) Use the graph of $v(t)$ to determine when the particle moves to the left and when it moves to the right.

(c) Find the location $s(t)$ of the particle at time t for $0 \le t \le 5$. Give a geometric interpretation of $s(t)$ in terms of the graph of $v(t)$.

(d) Graph $s(t)$ and find the leftmost and rightmost positions of the particle.

20. Recall that the acceleration $a(t)$ of a particle moving along a straight line is the instantaneous rate of change of the velocity $v(t)$; that is,

$$a(t) = \frac{d}{dt}v(t)$$

Assume that $a(t) = 32$ ft/s^2. Express the cumulative change in velocity during the interval $[0, t]$ as a definite integral, and compute the integral.

21. If $\frac{dl}{dt}$ represents the growth rate of an organism at time t (measured in months), explain what

$$\int_2^7 \frac{dl}{dt}\,dt$$

represents.

22. If $\frac{dw}{dx}$ represents the rate of change of the weight of an organism of age x, explain what

$$\int_3^5 \frac{dw}{dx}\,dx$$

means.

23. If $\frac{dB}{dt}$ represents the rate of change of biomass at time t, explain what

$$\int_1^6 \frac{dB}{dt}\,dt$$

means.

24. Let $N(t)$ denote the size of a population at time t, and assume that

$$\frac{dN}{dt} = f(t)$$

Express the cumulative change of the population size in the interval $[0, 3]$ as an integral.

■ 6.3.3

25. Let $f(x) = x^2 - 2$. Compute the average value of $f(x)$ over the interval $[0, 2]$.

26. Let $g(t) = \sin(\pi t)$. Compute the average value of $g(t)$ over the interval $[-1, 1]$.

27. Suppose that the temperature T (measured in degrees Fahrenheit) in a growing chamber varies over a 24-hour period according to

$$T(t) = 68 + \sin\left(\frac{\pi}{12}t\right)$$

for $0 \le t \le 24$.

(a) Graph the temperature T as a function of time t.

(b) Find the average temperature and explain your answer graphically.

28. Suppose that the concentration (measured in gm^{-3}) of nitrogen in the soil along a transect in moist tundra yields data points that follow a straight line with equation

$$y = 673.8 - 34.7x$$

for $0 \le x \le 10$, where x is the distance to the beginning of the transect. What is the average concentration of nitrogen in the soil along this transect?

29. Let $f(x) = \tan x$. Give a geometric argument to explain why the average value of $f(x)$ over $[-1, 1]$ is equal to 0.

30. Suppose that you drive from St. Paul to Duluth and you average 50 mph. Explain why there must be a time during your trip at which your speed is exactly 50 mph.

31. Let $f(x) = 2x, 0 \le x \le 2$. Use a geometric argument to find the average value of f over the interval $[0, 2]$, and find x such that $f(x)$ is equal to this average value.

32. A particle moves along the x-axis with velocity

$$v(t) = -(t-3)^2 + 5$$

for $0 \le t \le 6$.

(a) Graph $v(t)$ as a function of t for $0 \le t \le 6$.

(b) Find the average velocity of this particle during the interval $[0, 6]$.

(c) Find a time $t^* \in [0, 6]$ such that the velocity at time t^* is equal to the average velocity during the interval $[0, 6]$. Is it clear that such a point exists? Is there more than one such point in this case? Use your graph in (a) to explain how you would find t^* graphically.

■ 6.3.4

33. Find the volume of a right circular cone with base radius r and height h.

34. Find the volume of a pyramid with square base of side length a and height h.

In Problems 35–40, find the volumes of the solids obtained by rotating the region bounded by the given curves about the x-axis. In each case, sketch the region and a typical disk element.

35. $y = 4 - x^2, y = 0, x = 0$ (in the first quadrant)

36. $y = \sqrt{2x}, y = 0, x = 2$

37. $y = \sqrt{\sin x}, 0 \le x \le \pi, y = 0$

38. $y = e^x, y = 0, x = 0, x = \ln 2$

39. $y = \sec x, -\frac{\pi}{3} \le x \le \frac{\pi}{3}, y = 0$

40. $y = \sqrt{1 - x^2}, 0 \le x \le 1, y = 0$

In Problems 41–46, find the volumes of the solids obtained by rotating the region bounded by the given curves about the x-axis. In each case, sketch the region together with a typical disk element.

41. $y = x^2, y = x, 0 \le x \le 1$

42. $y = 2 - x^3, y = 2 + x^3, 0 \le x \le 1$

43. $y = e^x, y = e^{-x}, 0 \le x \le 2$

44. $y = \sqrt{1 - x^2}, y = 1, x = 1$ (in the first quadrant)

45. $y = \sqrt{\cos x}, y = 1, x = \frac{\pi}{2}$

46. $y = \frac{1}{x}, x = 0, y = 1, y = 2$ (in the first quadrant)

In Problems 47–52, find the volumes of the solids obtained by rotating the region bounded by the given curves about the y-axis. In each case, sketch the region together with a typical disk element.

47. $y = \sqrt{x}, y = 2, x = 0$

48. $y = x^2, y = 4, x = 0$ (in the first quadrant)

49. $y = \ln(x + 1), y = \ln 3, x = 0$

50. $y = \sqrt{x}, y = x, 0 \le x \le 1$

51. $y = x^2, y = \sqrt{x}, 0 \le x \le 1$

52. $y = \frac{1}{x}, x = 0, y = \frac{1}{2}, y = 1$

■ 6.3.5

53. Find the length of the straight line

$$y = 2x$$

from $x = 0$ to $x = 2$ by each of the following methods:

(a) planar geometry

(b) the integral formula for the lengths of curves, derived in Subsection 6.3.5

54. Find the length of the straight line

$$y = mx$$

from $x = 0$ to $x = a$, where m and a are positive constants, by each of the following methods:

(a) planar geometry

(b) the integral formula for the lengths of curves, derived in Subsection 6.3.5

55. Find the length of the curve

$$y^2 = x^3$$

from $x = 1$ to $x = 4$.

56. Find the length of the curve

$$2y^2 = 3x^3$$

from $x = 0$ to $x = 1$.

57. Find the length of the curve

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

from $x = 1$ to $x = 3$.

58. Find the length of the curve

$$y = \frac{x^4}{4} + \frac{1}{8x^2}$$

from $x = 2$ to $x = 4$.

In Problems 59–62, set up, but do not evaluate, the integrals for the lengths of the following curves:

59. $y = x^2, -1 \le x \le 1$

60. $y = \sin x, 0 \le x \le \frac{\pi}{2}$

61. $y = e^{-x}, 0 \le x \le 1$

62. $y = \ln x, 1 \le x \le e$

63. Find the length of the quarter-circle

$$y = \sqrt{1 - x^2}$$

for $0 \le x \le 1$, by each of the following methods:

(a) a formula from geometry

(b) the integral formula from Subsection 6.3.5

64. A cable that hangs between two poles at $x = -M$ and $x = M$ takes the shape of a catenary, with equation

$$y = \frac{1}{2a}(e^{ax} + e^{-ax})$$

where a is a positive constant. Compute the length of the cable when $a = 1$ and $M = \ln 2$.

65. Show that if

$$f(x) = \frac{e^x + e^{-x}}{2}$$

then the length of the curve $f(x)$ between $x = 0$ and $x = a$ for any $a > 0$ is given by $f'(a)$.

Chapter 6 Key Terms

Discuss the following definitions and concepts:

1. Area
2. Summation notation
3. Algebraic rules for sums
4. A partition of an interval and the norm of a partition
5. Riemann sum
6. Definite integral
7. Riemann integrable
8. Geometric interpretation of definite integrals
9. The constant-value and constant-multiple rules for integrals
10. The definite integral over a union of intervals
11. Comparison rules for definite integrals
12. The fundamental theorem of calculus, part I
13. Leibniz's rule
14. Antiderivatives
15. The fundamental theorem of calculus, part II
16. Evaluating definite integrals by using the FTC, part II
17. Computing the area between curves by using definite integrals
18. Cumulative change and definite integrals
19. The mean-value theorem for definite integrals
20. The volume of a solid and definite integrals
21. Rectification of curves
22. Length of a curve
23. Arc length differential

Chapter 6 Review Problems

1. **Discharge of a River** In studying the flow of water in an open channel, such as a river in its bed, the amount of water passing through a cross section per second—the discharge (Q)—is of interest. The following formula is used to compute the discharge:

$$Q = \int_0^B \overline{v}(b)h(b)\,db \tag{6.18}$$

In this formula, b is the distance from one bank of the river to the point where the depth $h(b)$ of the river and the average velocity $\overline{v}(b)$ of the vertical velocity profile of the river at b were measured. The total width of the cross section is B. (See Figure 6.48.)

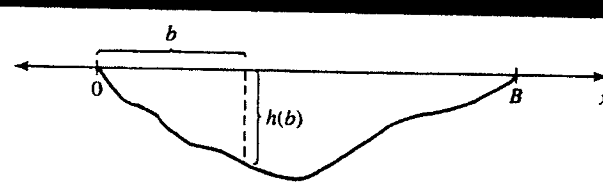

Figure 6.48 The river for Problem 1.

To evaluate the integral in (6.18), we would need to know $\overline{v}(b)$ and $h(b)$ at every location b along the cross section. In practice, the cross section is divided into a finite number of subintervals and measurements of $\overline{v}$ and h are taken at, say, the right endpoints of each subinterval. The following table contains an example of such measurements:

Location	*h*	$\bar{v}$
0	0	0
1	0.28	0.172
3	0.76	0.213
5	1.34	0.230
7	1.57	0.256
9	1.42	0.241
11	1.21	0.206
13	0.83	0.187
15	0.42	0.116
16	0	0

The location 0 corresponds to the left bank, and the location $B = 16$ to the right bank, of the river. The units of the location and of h are meters, and of $\bar{v}$, meters per second. Approximate the integral in (6.18) by a Riemann sum, using the locations in the table, and find the approximate discharge, using the data from the table.

2. Biomass Growth Suppose that you grow plants in several study plots and wish to measure the response of total biomass to the treatment in each plot. One way to measure this response would be to determine the average specific growth rate of the biomass for each plot over the course of the growing season.

We denote by $B(t)$ the biomass in a given plot at time t. Then the specific growth rate of the biomass at time t is given by

$$\frac{1}{B(t)}\frac{dB}{dt}$$

(a) Explain why

$$\frac{1}{t}\int_0^t \frac{1}{B(s)}\frac{dB(s)}{ds}\,ds$$

is a way to express the average specific growth rate over the interval $[0, t]$.

(b) Use the chain rule to show that

$$\frac{1}{B(t)}\frac{dB}{dt} = \frac{d}{dt}(\ln B(t))$$

(c) Use the results in (a) and (b) to show that the average specific growth rate of $B(s)$ over the interval $[0, t]$ is given by

$$\frac{1}{t}\int_0^t \frac{d}{ds}(\ln(B(s)))\,ds = \frac{1}{t}\ln\frac{B(t)}{B(0)}$$

provided that $B(s) > 0$ for $s \in [0, t]$.

(d) Explain the measurements that you would need to take if you wanted to determine the average specific growth rate of biomass in a given plot over the interval $[0, t]$.

Problems 3–6 discuss stream speed profiles and provide a justification for the two measurement methods described next.

(Adapted from Herschy, 1995) The speed of water in a channel varies considerably with depth. Due to friction, the speed reaches zero at the bottom and along the sides of the channel. The speed is greatest near the surface of the stream. To find the average speed for the vertical speed profile, two methods are frequently employed in practice:

1. The 0.6 depth method: The speed is measured at 0.6 of the depth from the surface, and this value is taken as the average speed.

2. The 0.2 and 0.8 depth method: The speed is measured at 0.2 and 0.8 of the depth from the surface, and the average of the two readings is taken as the average speed.

The theoretical speed distribution of water flowing in an open channel is given approximately by

$$v(d) = \left(\frac{D-d}{a}\right)^{1/c} \tag{6.19}$$

where $v(d)$ is the speed at depth d below the water surface, c is a constant varying from 5 for coarse beds to 7 for smooth beds, D is the total depth of the channel, and a is a constant that is equal to the distance above the bottom of the channel at which the speed has unit value.

3. (a) Sketch the graph of $v(d)$ as a function of d for $D = 3$ m and $a = 1$ m for (i) $c = 5$ and (ii) $c = 7$.

(b) Show that the speed is equal to 0 at the bottom ($d = D$) and is maximal at the surface ($d = 0$).

4. (a) Show by integration that the average speed $\bar{v}$ in the vertical profile is given by

$$\bar{v} = \frac{c}{c+1}\left(\frac{D}{a}\right)^{1/c} \tag{6.20}$$

(b) What fraction of the maximum speed is the average speed $\bar{v}$?

(c) If you knew that the maximum speed occurred at the surface of the river [as predicted in the approximate formula for $v(d)$], how could you find $\bar{v}$? (In practice, the maximum speed may occur quite a bit below the surface due to friction between the water on the surface and the atmosphere. Therefore, the speed at the surface would not be an accurate measure of the maximum speed.)

5. Explain why the depth d_1, at which $v =' \bar{v}$, is given by the equation

$$\bar{v} = \left(\frac{D-d_1}{a}\right)^{1/c} \tag{6.21}$$

We can find d_1 by equating (6.20) and (6.21). Show that

$$\frac{d_1}{D} = 1 - \left(\frac{c}{c+1}\right)^c$$

and that d_1/D is approximately 0.6 for values of c between 5 and 7, thus resulting in the rule

$$\bar{v} \approx v_{0.6}$$

where $v_{0.6}$ is the speed at depth $0.6D$. (*Hint*: Graph $1-(c/(c+1))^c$ as a function of c for $c \in [5, 7]$, and investigate the range of this function.)

6. We denote by $v_{0.2}$ the speed at depth $0.2D$. We will now find the depth d_2 such that

$$\bar{v} = \tfrac{1}{2}(v_{0.2} + v_{d_2})$$

(a) Show that d_2 satisfies

$$\frac{1}{2}\left[\left(\frac{D-0.2D}{a}\right)^{1/c} + \left(\frac{D-d_2}{a}\right)^{1/c}\right] = \frac{c}{c+1}\left(\frac{D}{a}\right)^{1/c}$$

[*Hint*: Use (6.19) and (6.20).]

(b) Show that

$$\frac{d_2}{D} = 1 - \left[\frac{2c}{c+1} - (0.8)^{1/c}\right]^c$$

and confirm that d_2/D is approximately 0.8 for values of c between 5 and 7, thus resulting in the rule

$$\bar{v} \approx \tfrac{1}{2}(v_{0.2} + v_{0.8})$$