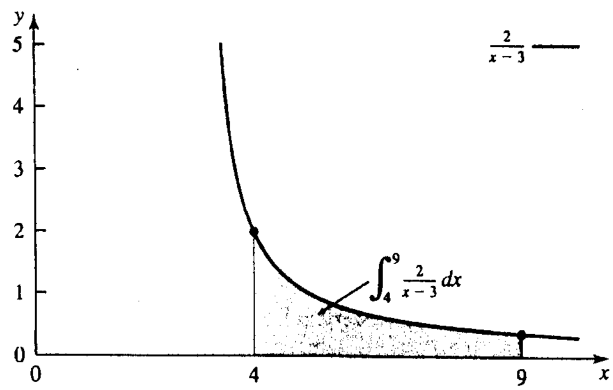

Figure 7.6 The region corresponding to the definite integral in Example 10.

Therefore,

$$\int_4^9 \frac{2}{x-3}\,dx = \int_1^6 \frac{2}{u}\,du = 2\ln|u|]_1^6 = 2(\ln 6 - \ln 1) = 2\ln 6$$ ■

We can easily spend a great deal of time on integration techniques. The problems can get very involved, and to solve them all, we would need a big bag full of tricks. There are excellent software programs (such as *Mathematica* and MATLAB®) that can integrate symbolically. These programs do not render integration techniques useless; in fact, they use them. Understanding the basic techniques conceptually and being able to apply them in simple situations makes such software packages less of a "black box." Nevertheless, their availability has made it less important to acquire a large number of tricks.

So far, we have learned only one technique: substitution. Unless you can immediately recognize an antiderivative, substitution is the only method you can try at this point.

As we proceed, you will learn other techniques. An additional complication will then be to recognize which technique to use. If you don't see right away what to do, just try something. Don't always expect the first attempt to succeed. With practice, you will see much more quickly whether or not your approach will succeed. If your attempt does not seem to work, try to determine the reason. That way, failed attempts can be quite useful for gaining experience in integration.

Section 7.1 Problems

■ 7.1.1

In Problems 1–16, evaluate the indefinite integral by making the given substitution.

1. $\int 2x\sqrt{x^2+3}\,dx$, with $u = x^2+3$

2. $\int 3x^2\sqrt{x^3+1}\,dx$, with $u = x^3+1$

3. $\int 3x(1-x^2)^{1/4}\,dx$, with $u = 1-x^2$

4. $\int 4x^3(4+x^4)^{1/3}\,dx$, with $u = 4+x^4$

5. $\int 5\cos(3x)\,dx$, with $u = 3x$

6. $\int 5\sin(1-2x)\,dx$, with $u = 1-2x$

7. $\int 7x^2\sin(4x^3)\,dx$, with $u = 4x^3$

8. $\int x\cos(x^2-1)\,dx$, with $u = x^2-1$

9. $\int e^{2x+3}\,dx$, with $u = 2x+3$

10. $\int 3e^{1-x}\,dx$, with $u = 1-x$

11. $\int xe^{-x^2/2}\,dx$, with $u = -x^2/2$

12. $\int xe^{1-3x^2}\,dx$, with $u = 1-3x^2$

13. $\int \frac{x+2}{x^2+4x}\,dx$, with $u = x^2 + 4x$

14. $\int \frac{2x}{3-x^2}\,dx$, with $u = 3 - x^2$

15. $\int \frac{3x}{x+4}\,dx$, with $u = x + 4$

16. $\int \frac{x}{5-x}\,dx$, with $u = 5 - x$

In Problems 17–36, use substitution to evaluate the indefinite integrals.

17. $\int \sqrt{x+3}\,dx$

18. $\int (4-x)^{1/7}\,dx$

19. $\int (4x-3)\sqrt{2x^2-3x+2}\,dx$

20. $\int (x^2-2x)(x^3-3x^2+3)^{2/3}\,dx$

21. $\int \frac{x-1}{1+4x-2x^2}\,dx$

22. $\int \frac{x^2-1}{x^3-3x+1}\,dx$

23. $\int \frac{2x}{1+2x^2}\,dx$

24. $\int \frac{x^3-1}{x^4-4x}\,dx$

25. $\int 3xe^{x^2}\,dx$

26. $\int \cos x e^{\sin x}\,dx$

27. $\int \frac{1}{x}\csc^2(\ln x)\,dx$

28. $\int \sec^2 x e^{\tan x}\,dx$

29. $\int \sin\left(\frac{3\pi}{2}x + \frac{\pi}{4}\right)\,dx$

30. $\int \cos(2x-1)\,dx$

31. $\int \tan x \sec^2 x\,dx$

32. $\int \sin^3 x \cos x\,dx$

33. $\int \frac{(\ln x)^2}{x}\,dx$

34. $\int \frac{dx}{(x-3)\ln(x-3)}$

35. $\int x^3\sqrt{5+x^2}\,dx$

36. $\int \sqrt{1+\ln x}\frac{\ln x}{x}\,dx$

In Problems 37–42, a, b, and c are constants and g(x) is a continuous function whose derivative g′(x) is also continuous. Use substitution to evaluate the indefinite integrals.

37. $\int \frac{2ax+b}{ax^2+bx+c}\,dx$

38. $\int \frac{1}{ax+b}\,dx$

39. $\int g'(x)[g(x)]^n\,dx$

40. $\int g'(x)\sin[g(x)]\,dx$

41. $\int g'(x)e^{-g(x)}\,dx$

42. $\int \frac{g'(x)}{[g(x)]^2+1}\,dx$

7.1.2

In Problems 43–58, use substitution to evaluate the definite integrals.

43. $\int_0^3 x\sqrt{x^2+1}\,dx$

44. $\int_1^2 x^5\sqrt{x^3+2}\,dx$

45. $\int_2^3 \frac{2x+3}{(x^2+3x)^3}\,dx$

46. $\int_0^2 \frac{2x}{(4x^2+3)^{1/3}}\,dx$

47. $\int_2^5 (x-2)e^{-(x-2)^2/2}\,dx$

48. $\int_{\ln 4}^{\ln 7} \frac{e^x}{(e^x-3)^2}\,dx$

49. $\int_0^{\pi/3} \sin x\cos x\,dx$

50. $\int_{-\pi/6}^{\pi/6} \sin^2 x\cos x\,dx$

51. $\int_0^{\pi/4} \tan x\sec^2 x\,dx$

52. $\int_0^{\pi/3} \frac{\sin x}{\cos^2 x}\,dx$

53. $\int_5^9 \frac{x}{x-3}\,dx$

54. $\int_0^2 \frac{x}{x+2}\,dx$

55. $\int_e^{e^2} \frac{dx}{x(\ln x)^2}$

56. $\int_1^2 \frac{x\,dx}{(x^2+1)\ln(x^2+1)}$

57. $\int_1^9 \frac{1}{\sqrt{x}}e^{-\sqrt{x}}\,dx$

58. $\int_0^2 x\sqrt{4-x^2}\,dx$

59. Use the fact that

$$\cot x = \frac{\cos x}{\sin x}$$

to evaluate

$$\int \cot x\,dx$$

7.2 Integration by Parts and Practicing Integration

7.2.1 Integration by Parts

As mentioned at the beginning of this chapter, integration by parts is the product rule in integral form. Let $u = u(x)$ and $v = v(x)$ be differentiable functions. Then, differentiating with respect to x yields

$$(uv)' = u'v + uv'$$

or, after rearranging,

$$uv' = (uv)' - u'v$$

Integrating both sides with respect to x, we find that

$$\int uv'\,dx = \int (uv)'\,dx - \int u'v\,dx$$

Since uv is an antiderivative of $(uv)'$, it follows that

$$\int (uv)'\,dx = uv + C$$

Combining our results, we find that

$$\int x^{1/2}\ln(x^{1/2}e^x)\,dx = \frac{1}{2}\int x^{1/2}\ln x\,dx + \int x^{3/2}dx$$
$$= \frac{1}{3}x^{3/2}\left(\ln x - \frac{2}{3}\right) + \frac{2}{5}x^{5/2} + C$$

Note that we used the same symbol C to denote the integration constants. We could have called them C_1 and C_2 and then combined them into $C = C_1 + C_2$, but since they stand for arbitrary constants, we need not keep track of how they are related and can simply capture them all by the same symbol. However, we should keep in mind that they are not all the same. ■

Section 7.2 Problems

■ 7.2.1

In Problems 1–30, use integration by parts to evaluate the integrals.

1. $\int x\cos x\,dx$
2. $\int 3x\cos x\,dx$
3. $\int 2x\cos(3x-1)\,dx$
4. $\int 3x\cos(4-x)\,dx$
5. $\int 2x\sin(x-1)\,dx$
6. $\int x\sin(1-2x)\,dx$
7. $\int xe^x\,dx$
8. $\int 3xe^{-x/2}\,dx$
9. $\int x^2e^x\,dx$
10. $\int 2x^2e^{-x}\,dx$
11. $\int x\ln x\,dx$
12. $\int x^2\ln x\,dx$
13. $\int x\ln(3x)\,dx$
14. $\int x^2\ln x^2\,dx$
15. $\int x\sec^2 x\,dx$
16. $\int x\csc^2 x\,dx$
17. $\int_0^{\pi/3} x\sin x\,dx$
18. $\int_0^{\pi/4} 2x\cos x\,dx$
19. $\int_1^2 \ln x\,dx$
20. $\int_1^e \ln x^2\,dx$
21. $\int_1^4 \ln\sqrt{x}\,dx$
22. $\int_1^4 \sqrt{x}\ln\sqrt{x}\,dx$
23. $\int_0^1 xe^{-x}\,dx$
24. $\int_0^3 x^2e^{-x}\,dx$
25. $\int_0^{\pi/3} e^x\sin x\,dx$
26. $\int_0^{\pi/6} e^x\cos x\,dx$
27. $\int e^{-3x}\cos\left(\frac{\pi}{2}x\right)dx$
28. $\int e^{-2x}\sin\left(\frac{x}{2}\right)dx$
29. $\int \sin(\ln x)\,dx$
30. $\int \cos(\ln x)\,dx$

31. Evaluating the integral

$$\int \cos^2 x\,dx$$

requires two steps.

First, write

$$\cos^2 x = (\cos x)(\cos x)$$

and integrate by parts to show that

$$\int \cos^2 x\,dx = \sin x\cos x + \int \sin^2 x\,dx$$

Then, use $\sin^2 x + \cos^2 x = 1$ to replace $\sin^2 x$ in the integral on the right-hand side, and complete the integration of $\int \cos^2 x\,dx$.

32. Evaluating the integral

$$\int \sin^2 x\,dx$$

requires two steps.

First, write

$$\sin^2 x = (\sin x)(\sin x)$$

and integrate by parts to show that

$$\int \sin^2 x\,dx = -\sin x\cos x + \int \cos^2 x\,dx$$

Then, use $\sin^2 x + \cos^2 x = 1$ to replace $\cos^2 x$ in the integral on the right-hand side, and complete the integration of $\int \sin^2 x\,dx$.

33. Evaluating the integral

$$\int \arcsin x\,dx$$

requires two steps.

(a) Write

$$\arcsin x = 1\cdot\arcsin x$$

and integrate by parts once to show that

$$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

(b) Use substitution to compute

$$\int \frac{x}{\sqrt{1-x^2}}\,dx \qquad (7.7)$$

and combine your result in (a) with (7.7) to complete the computation of $\int \arcsin x\,dx$.

34. Evaluating the integral

$$\int \arccos x\, dx$$

requires two steps.
(a) Write

$$\arccos x = 1 \cdot \arccos x$$

and integrate by parts once to show that

$$\int \arccos x\, dx = x \arccos x + \int \frac{x}{\sqrt{1-x^2}}\, dx$$

(b) Use substitution to compute

$$\int \frac{x}{\sqrt{1-x^2}}\, dx \qquad (7.8)$$

and combine your result in (a) with (7.8) to complete the computation of $\int \arccos x\, dx$.

35. (a) Use integration by parts to show that, for $x > 0$,

$$\int \frac{1}{x} \ln x\, dx = (\ln x)^2 - \int \frac{1}{x} \ln x\, dx$$

(b) Use your result in (a) to evaluate

$$\int \frac{1}{x} \ln x\, dx$$

36. (a) Use integration by parts to show that

$$\int x^n e^x\, dx = x^n e^x - n \int x^{n-1} e^x\, dx$$

Such formulas are called **reduction formulas**, since they reduce the exponent of x by 1 each time they are applied.
(b) Apply the reduction formula in (a) repeatedly to compute

$$\int x^3 e^x\, dx$$

37. (a) Use integration by parts to verify the validity of the reduction formula

$$\int x^n e^{ax}\, dx = \frac{1}{a} x^n e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax}\, dx$$

where a is a constant not equal to 0.
(b) Apply the reduction formula in (a) to compute

$$\int x^2 e^{-3x}\, dx$$

38. (a) Use integration by parts to verify the validity of the reduction formula

$$\int (\ln x)^n\, dx = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx$$

(b) Apply the reduction formula in (a) repeatedly to compute

$$\int (\ln x)^3\, dx$$

In Problems 39–48, first make an appropriate substitution and then use integration by parts to evaluate the indefinite integrals.

39. $\int \cos \sqrt{x}\, dx$ 40. $\int \sin \sqrt{x}\, dx$

41. $\int x^3 e^{-x^2/2}\, dx$ 42. $\int x^5 e^{x^2}\, dx$

43. $\int \sin x \cos x\, e^{\sin x}\, dx$ 44. $\int \sin x \cos^3 x\, e^{1-\sin^2 x}\, dx$

45. $\int_0^1 e^{\sqrt{x}}\, dx$ 46. $\int_1^2 e^{\sqrt{x+1}}\, dx$

47. $\int_1^4 \ln(\sqrt{x}+1)\, dx$ 48. $\int_0^1 x^3 \ln(x^2+1)\, dx$

■ 7.2.2

In Problems 49–60, use either substitution or integration by parts to evaluate each integral.

49. $\int x e^{-2x}\, dx$ 50. $\int x e^{-2x^2}\, dx$

51. $\int \frac{1}{\tan x}\, dx$ 52. $\int \frac{1}{\csc x \sec x}\, dx$

53. $\int 2x \sin(x^2)\, dx$ 54. $\int 2x^2 \sin x\, dx$

55. $\int \frac{1}{16+x^2}\, dx$ 56. $\int \frac{1}{x^2+5}\, dx$

57. $\int \frac{x}{x+3}\, dx$ 58. $\int \frac{1}{x^2+3}\, dx$

59. $\int \frac{x}{x^2+3}\, dx$ 60. $\int \frac{x+2}{x^2+2}\, dx$

61. The integral

$$\int \ln x\, dx$$

can be evaluated in two ways.
(a) Write $\ln x = 1 \cdot \ln x$ and use integration by parts to evaluate the integral.
(b) Use the substitution $u = \ln x$ and integration by parts to evaluate the integral.

62. Use an appropriate substitution followed by integration by parts to evaluate

$$\int x^3 e^{-x^2/2}\, dx$$

63. Use an appropriate substitution to evaluate

$$\int x(x-2)^{1/4}\, dx$$

64. Simplify the integrand and then use an appropriate substitution to evaluate

$$\int \frac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}\, dx$$

In Problems 65–70, evaluate each definite integral.

65. $\int_1^4 e^{\sqrt{x}}\, dx$ 66. $\int_1^2 \ln(x^2 e^x)\, dx$

67. $\int_{-1}^0 \frac{2}{1+x^2}\, dx$ 68. $\int_1^2 x^2 \ln x\, dx$

69. $\int_0^{\pi/4} e^x \sin x\, dx$ 70. $\int_0^{\pi/6} (1+\tan^2 x)\, dx$

We conclude this section by providing a summary of the two most important cases: when the integrand is a rational function for which the denominator is a polynomial of degree 2 and is either (1) a product of two not necssarily distinct linear factors or (2) an irreducible quadratic polynomial.

The first step is to make sure that the degree of the numerator is less than the degree of the denominator. If not, then we use long division to simplify the integrand.

We will now assume that the degree of the numerator is strictly less than the degree of the denominator (i.e., the integrand is a proper rational function). We write the rational function $f(x)$ as

$$f(x) = \frac{P(x)}{Q(x)}$$

with $Q(x) = ax^2 + bx + c, a \neq 0$, and $P(x) = rx + s$. Either $Q(x)$ can be factored into two linear factors, or it is irreducible (i.e., does not have real roots).

Case 1a: $Q(x)$ is a product of two distinct linear factors. In this case, we write

$$Q(x) = a(x - x_1)(x - x_2)$$

where x_1 and x_2 are the two distinct roots of $Q(x)$. We then use the method of partial fractions to simplify the rational function:

$$\frac{P(x)}{Q(x)} = \frac{rx + s}{ax^2 + bx + c} = \frac{1}{a}\left[\frac{A}{x - x_1} + \frac{B}{x - x_2}\right]$$

The constants A and B must now be determined as in Example 3.

Case 1b: $Q(x)$ is a product of two identical linear factors. In this case, we write

$$Q(x) = a(x - x_1)^2$$

where x_1 is the root of $Q(x)$. We then use the method of partial fractions to simplify the rational function:

$$\frac{P(x)}{Q(x)} = \frac{rx + s}{ax^2 + bx + c} = \frac{1}{a}\left[\frac{A}{x - x_1} + \frac{B}{(x - x_1)^2}\right]$$

The constants A and B must now be determined as in Example 5.

Case 2: $Q(x)$ is an irreducible quadratic polynomial. In this case,

$$Q(x) = ax^2 + bx + c \quad \text{with } b^2 - 4ac < 0$$

and we must complete the square as in Example 2. Doing so then leads to integrals of the form

$$\int \frac{dx}{x^2 + 1} \quad \text{or} \quad \int \frac{x}{x^2 + 1}\,dx$$

The first integral is $\tan^{-1} x + C$, whereas the second integral can be evaluated by substitution. (See Examples 6 and 7.)

Section 7.3 Problems

■ **7.3.1, 7.3.2**

In Problems 1–4, use long division to write $f(x)$ as a sum of a polynomial and a proper rational function.

1. $f(x) = \dfrac{2x^2 + 5x - 1}{x + 2}$

2. $f(x) = -\dfrac{x^2 - 4x - 1}{x - 1}$

3. $f(x) = \dfrac{3x^3 + 5x - 2x^2 - 2}{x^2 + 1}$

4. $f(x) = \dfrac{x^3 - 3x^2 - 15}{x^2 + x + 3}$

In Problems 5–8, write out the partial-fraction decomposition of the function $f(x)$.

5. $f(x) = \dfrac{2x - 3}{x(x + 1)}$

6. $f(x) = -\dfrac{x + 1}{(2x + 1)(x - 1)}$

7. $f(x) = \dfrac{4x^2 - 14x - 6}{x(x - 3)(x + 1)}$

8. $f(x) = \dfrac{16x - 6}{(2x - 5)(3x + 1)}$

In Problems 9–12, write out the partial-fraction decomposition of the function $f(x)$.

9. $f(x) = \dfrac{5x-1}{x^2-1}$

10. $f(x) = \dfrac{9x-7}{2x^2-7x+3}$

11. $f(x) = \dfrac{4x+1}{x^2-3x-10}$

12. $f(x) = -\dfrac{10}{3x^2+8x-3}$

In Problems 13–18, use partial-fraction decomposition to evaluate the integrals.

13. $\displaystyle\int \frac{1}{x(x-2)}\,dx$

14. $\displaystyle\int \frac{1}{x(2x+1)}\,dx$

15. $\displaystyle\int \frac{1}{(x+1)(x-3)}\,dx$

16. $\displaystyle\int \frac{1}{(x-1)(x+2)}\,dx$

17. $\displaystyle\int \frac{x^2-2x-2}{x^2(x+2)}\,dx$

18. $\displaystyle\int \frac{4x^2-x-1}{(x+1)^2(x-3)}\,dx$

In Problems 19–22, use partial-fraction decompositon to evaluate each integral.

19. $\displaystyle\int \frac{x^3-x^2+x-4}{(x^2+1)(x^2+4)}\,dx$

20. $\displaystyle\int \frac{x^3-3x^2+x-6}{(x^2+2)(x^2+1)}\,dx$

21. $\displaystyle\int \frac{2x^2-3x+2}{(x^2+1)^2}\,dx$

22. $\displaystyle\int \frac{3x^2+4x+3}{(x^2+1)^2}\,dx$

In Problems 23–26, complete the square in the denominator and evaluate the integral.

23. $\displaystyle\int \frac{1}{x^2-2x+2}\,dx$

24. $\displaystyle\int \frac{1}{x^2+4x+5}\,dx$

25. $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$

26. $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$

In Problems 27–36, evaluate each integral.

27. $\displaystyle\int \frac{1}{(x-3)(x+2)}\,dx$

28. $\displaystyle\int \frac{2x-1}{(x+4)(x+1)}\,dx$

29. $\displaystyle\int \frac{1}{x^2-9}\,dx$

30. $\displaystyle\int \frac{1}{x^2+9}\,dx$

31. $\displaystyle\int \frac{1}{x^2-x-2}\,dx$

32. $\displaystyle\int \frac{1}{x^2-x+2}\,dx$

33. $\displaystyle\int \frac{x^2+1}{x^2+3x+2}\,dx$

34. $\displaystyle\int \frac{x^3+1}{x^2+3}\,dx$

35. $\displaystyle\int \frac{x^2+4}{x^2-4}\,dx$

36. $\displaystyle\int \frac{x^4+3}{x^2-4x+3}\,dx$

In Problems 37–44, evaluate each definite integral.

37. $\displaystyle\int_3^5 \frac{x-1}{x}\,dx$

38. $\displaystyle\int_3^5 \frac{x}{x-1}\,dx$

39. $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$

40. $\displaystyle\int_1^2 \frac{x^2+1}{x}\,dx$

41. $\displaystyle\int_2^3 \frac{1}{1-x}\,dx$

42. $\displaystyle\int_2^3 \frac{1}{1-x^2}\,dx$

43. $\displaystyle\int_0^1 \tan^{-1} x\,dx$

44. $\displaystyle\int_0^1 x\tan^{-1} x\,dx$

In Problems 45–52, evaluate each integral.

45. $\displaystyle\int \frac{1}{(x+1)^2 x}\,dx$

46. $\displaystyle\int \frac{1}{x^2(x-1)^2}\,dx$

47. $\displaystyle\int \frac{4}{(1-x)(1+x)^2}\,dx$

48. $\displaystyle\int \frac{2x^2+2x-1}{x^3(x-3)}\,dx$

49. $\displaystyle\int \frac{1}{(x^2-9)^2}\,dx$

50. $\displaystyle\int \frac{1}{(x^2-x-2)^2}\,dx$

51. $\displaystyle\int \frac{1}{x^2(x^2+1)}\,dx$

52. $\displaystyle\int \frac{1}{(x+1)^2(x^2+1)}\,dx$

53. **(a)** To complete Example 8, show that

$$\frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

(b) Show that

$$\int_0^1 \frac{x^4(1-x)^4}{2}\,dx \le \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx \le \int_0^1 x^4(1-x)^4\,dx$$

and conclude that

$$\frac{1}{1260} \le \frac{22}{7} - \pi \le \frac{1}{630}$$

Use this result to show that

$$3.140 \le \pi \le 3.142$$

■ 7.4 Improper Integrals

In this section, we discuss definite integrals of two types with the following characteristics:

1. One or both limits of integration are infinite; that is, the integration interval is unbounded; or
2. The integrand becomes infinite at one or more points of the interval of integration.

We call such integrals **improper integrals**.

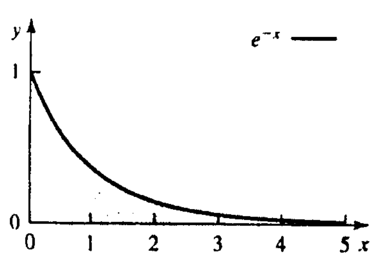

Figure 7.9 The unbounded region between the graph of $y = e^{-x}$ and the x-axis for $x \ge 0$.

■ 7.4.1 Type 1: Unbounded Intervals

Suppose that we wanted to compute the area of the unbounded region below the graph of $f(x) = e^{-x}$ and above the x-axis for $x \ge 0$. (See Figure 7.9.) How would we proceed? We know how to find the area of a region bounded by the graph of a continuous function [here, $f(x) = e^{-x}$] and the x-axis between 0 and z, namely,

$$A(z) = \int_0^z e^{-x}\,dx = -e^{-x}\Big]_0^z = 1 - e^{-z}$$

Solution The function $f(x) = 1/\sqrt{x+\sqrt{x}}$ is continuous on $[1, \infty)$. The integrand looks rather complicated, but since $x + \sqrt{x} \le x + x$ for $x \ge 1$, it follows that

$$\frac{1}{\sqrt{x+\sqrt{x}}} \ge \frac{1}{\sqrt{2x}} \quad \text{for } x \ge 1$$

Hence,

$$\int_1^\infty \frac{1}{\sqrt{x+\sqrt{x}}}\,dx \ge \int_1^\infty \frac{1}{\sqrt{2x}}\,dx = \frac{1}{\sqrt{2}}\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \infty$$

as shown in Example 2. Therefore,

$$\int_1^\infty \frac{dx}{\sqrt{x+\sqrt{x}}}$$

is divergent. ■

Section 7.4 Problems

■ **7.4.1, 7.4.2**

All the integrals in Problems 1–16 are improper and converge. Explain in each case why the integral is improper, and evaluate each integral.

1. $\int_0^\infty 3e^{-6x}\,dx$
2. $\int_0^\infty xe^{-x}\,dx$
3. $\int_0^\infty \frac{2}{1+x^2}\,dx$
4. $\int_e^\infty \frac{dx}{x(\ln x)^2}$
5. $\int_1^\infty \frac{1}{x^{3/2}}\,dx$
6. $\int_{-\infty}^{-1} \frac{1}{1+x^2}\,dx$
7. $\int_{-\infty}^\infty e^{-|x|}\,dx$
8. $\int_{-\infty}^\infty xe^{-x^2/2}\,dx$
9. $\int_{-\infty}^\infty \frac{x}{(1+x^2)^2}\,dx$
10. $\int_{-\infty}^\infty x^3e^{-x^4}\,dx$
11. $\int_0^9 \frac{dx}{\sqrt{9-x}}$
12. $\int_1^e \frac{dx}{x\sqrt{\ln x}}$
13. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{\sin x}}\,dx$
14. $\int_{-2}^0 \frac{dx}{(x+1)^{1/3}}$
15. $\int_{-1}^1 \ln|x|\,dx$
16. $\int_0^2 \frac{dx}{(x-1)^{2/5}}$

In Problems 17–28, determine whether each integral is convergent. If the integral is convergent, compute its value.

17. $\int_1^\infty \frac{1}{x^3}\,dx$
18. $\int_1^\infty \frac{1}{x^{1/3}}\,dx$
19. $\int_0^4 \frac{1}{x^4}\,dx$
20. $\int_0^4 \frac{1}{x^{1/4}}\,dx$
21. $\int_0^2 \frac{1}{(x-1)^{1/3}}\,dx$
22. $\int_0^2 \frac{1}{(x-1)^4}\,dx$
23. $\int_0^\infty \frac{1}{\sqrt{x+1}}\,dx$
24. $\int_{-1}^0 \frac{1}{\sqrt{x+1}}\,dx$
25. $\int_e^\infty \frac{dx}{x\ln x}$
26. $\int_1^e \frac{dx}{x\ln x}$
27. $\int_{-2}^2 \frac{2x\,dx}{(x^2-1)^{1/3}}$
28. $\int_{-\infty}^1 \frac{3}{1+x^2}\,dx$
29. Determine whether

$$\int_{-\infty}^\infty \frac{1}{x^2-1}\,dx$$

is convergent. *Hint*: Use the partial-fraction decomposition

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

30. Although we cannot compute the antiderivative of $f(x) = e^{-x^2/2}$, it is known that

$$\int_{-\infty}^\infty e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Use this fact to show that

$$\int_{-\infty}^\infty x^2e^{-x^2/2}\,dx = \sqrt{2\pi}$$

Hint: Write the integrand as

$$x \cdot (xe^{-x^2/2})$$

and use integration by parts.

31. Determine the constant c so that

$$\int_0^\infty ce^{-3x}\,dx = 1$$

32. Determine the constant c so that

$$\int_{-\infty}^\infty \frac{c}{1+x^2}\,dx = 1$$

33. In this problem, we investigate the integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

for $0 < p < \infty$.

(a) For $z > 1$, set

$$A(z) = \int_1^z \frac{1}{x^p}\,dx$$

and show that

$$A(z) = \frac{1}{1-p}(z^{-p+1} - 1)$$

for $p \neq 1$ and

$$A(z) = \ln z$$

for $p = 1$.

(b) Use your results in (a) to show that, for $0 < p \leq 1$,

$$\lim_{z\to\infty} A(z) = \infty$$

(c) Use your results in (a) to show that, for $p > 1$,

$$\lim_{z\to\infty} A(z) = \frac{1}{p-1}$$

34. In this problem, we investigate the integral

$$\int_0^1 \frac{1}{x^p}\,dx$$

for $0 < p < \infty$.

(a) Compute

$$\int \frac{1}{x^p}\,dx$$

for $0 < p < \infty$. (*Hint:* Treat the case where $p = 1$ separately.)

(b) Use your result in (a) to compute

$$\int_c^1 \frac{1}{x^p}\,dx$$

for $0 < c < 1$.

(c) Use your result in (b) to show that

$$\int_0^1 \frac{1}{x^p}\,dx = \frac{1}{1-p}$$

for $0 < p < 1$.

(d) Show that

$$\int_0^1 \frac{1}{x^p}\,dx$$

is divergent for $p \geq 1$.

■ 7.4.3

35. (a) Show that

$$0 \leq e^{-x^2} \leq e^{-x}$$

for $x \geq 1$.

(b) Use your result in (a) to show that

$$\int_1^\infty e^{-x^2}\,dx$$

is convergent.

36. (a) Show that

$$0 \leq \frac{1}{\sqrt{1+x^4}} \leq \frac{1}{x^2}$$

for $x > 0$.

(b) Use your result in (a) to show that

$$\int_1^\infty \frac{1}{\sqrt{1+x^4}}\,dx$$

is convergent.

37. (a) Show that

$$\frac{1}{\sqrt{1+x^2}} \geq \frac{1}{2x} > 0$$

for $x \geq 1$.

(b) Use your result in (a) to show that

$$\int_1^\infty \frac{1}{\sqrt{1+x^2}}\,dx$$

is divergent.

38. (a) Show that

$$\frac{1}{\sqrt{x+\ln x}} \geq \frac{1}{\sqrt{2x}} > 0$$

for $x \geq 1$.

(b) Use your result in (a) to show that

$$\int_1^\infty \frac{1}{\sqrt{x+\ln x}}\,dx$$

is divergent.

In Problems 39–42, find a comparison function for each integrand and determine whether the integral is convergent.

39. $\displaystyle\int_{-\infty}^\infty e^{-x^2/2}\,dx$

40. $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x^6}}\,dx$

41. $\displaystyle\int_1^\infty \frac{1}{\sqrt{1+x}}\,dx$

42. $\displaystyle\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$

43. (a) Show that

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}} = 0$$

(b) Use your result in (a) to show that

$$2\ln x \leq \sqrt{x} \qquad (7.17)$$

for sufficiently large x. Use a graphing calculator to determine just how large x must be for (7.17) to hold.

(c) Use your result in (b) to show that

$$\int_0^\infty e^{-\sqrt{x}}\,dx \qquad (7.18)$$

converges. Use a graphing calculator to sketch the function $f(x) = e^{-\sqrt{x}}$ together with its comparison function(s), and use your graph to explain how you showed that the integral in (7.18) is convergent.

44. (a) Show that

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0$$

(b) Use your result in (a) to show that, for any $c > 0$,

$$cx \geq \ln x$$

for sufficiently large x.

(c) Use your result in (b) to show that, for any $p > 0$,

$$x^p e^{-x} \leq e^{-x/2}$$

provided that x is sufficiently large.

(d) Use your result in (c) to show that, for any $p > 0$,

$$\int_0^\infty x^p e^{-x}\,dx$$

is convergent.

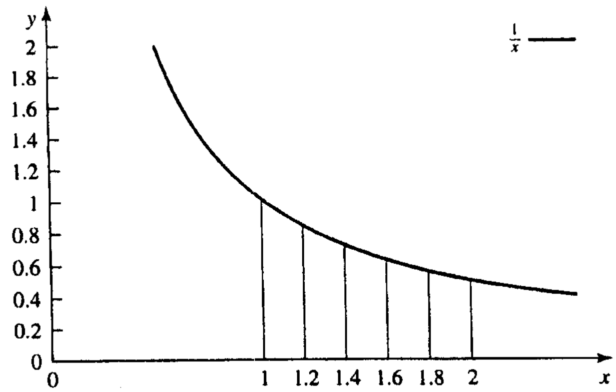

Figure 7.39 The trapezoidal rule for $\int_1^2 \frac{1}{x}\,dx$ with $n = 5$.

In Example 3, since $f(x) = x^2$, it follows that $f''(x) = 2$ and hence $K = 2$. The error is therefore bounded by

$$2\frac{1}{12(4^2)} = 0.0104$$

which is the same as the actual error.

In Example 4, since $f(x) = 1/x$, we have $|f''(x)| = 2/x^3 \leq 2$ for $1 \leq x \leq 2$ (as in Example 2). Hence, with $n = 5$, the error bound is at most

$$2\frac{(2-1)^3}{12(5^2)} = 0.0067$$

The actual error was in fact smaller, only 0.00249. As with the midpoint rule, the theoretical error can be quite a bit larger than the actual error.

Section 7.5 Problems

■ 7.5.1, 7.5.2

In Problems 1–4, use the midpoint rule to approximate each integral with the specified value of n.

1. $\int_1^2 x^2\,dx, n = 4$

2. $\int_{-1}^0 (x+1)^3\,dx, n = 5$

3. $\int_0^1 e^{-x}\,dx, n = 3$

4. $\int_0^{\pi/2} \sin x\,dx, n = 4$

In Problems 5–8, use the midpoint rule to approximate each integral with the specified value of n. Compare your approximation with the exact value.

5. $\int_2^4 \frac{1}{x}\,dx, n = 4$

6. $\int_{-1}^1 (e^{2x} - 1)\,dx, n = 4$

7. $\int_0^4 \sqrt{x}\,dx, n = 4$

8. $\int_2^4 \frac{2}{\sqrt{x}}\,dx, n = 5$

In Problems 9–12, use the trapezoidal rule to approximate each integral with the specified value of n.

9. $\int_1^2 x^2\,dx, n = 4$

10. $\int_{-1}^0 x^3\,dx, n = 5$

11. $\int_0^1 e^{-x}\,dx, n = 3$

12. $\int_0^{\pi/2} \sin x\,dx, n = 4$

In Problems 13–16, use the trapezoidal rule to approximate each integral with the specified value of n. Compare your approximation with the exact value.

13. $\int_1^3 x^3\,dx, n = 5$

14. $\int_{-1}^1 (1 - e^{-x})\,dx, n = 4$

15. $\int_0^2 \sqrt{x}\,dx, n = 4$

16. $\int_1^2 \frac{1}{x}\,dx, n = 5$

17. How large should n be so that the midpoint rule approximation of

$$\int_0^2 x^2\,dx$$

is accurate to within 10^{-4}?

In Problems 18–24, use the theoretical error bound to determine how large n should be. [Hint: In each case, find the second derivative of the integrand, graph it, and use a graphing calculator to find an upper bound on $|f''(x)|$.]

18. How large should n be so that the midpoint rule approximation of

$$\int_1^2 \frac{1}{x}\,dx$$

is accurate to within 10^{-3}?

19. How large should n be so that the midpoint rule approximation of
$$\int_0^2 e^{-x^2/2}\,dx$$
is accurate to within 10^{-4}?

20. How large should n be so that the midpoint rule approximation of
$$\int_2^8 \frac{1}{\ln t}\,dt$$
is accurate to within 10^{-3}?

21. How large should n be so that the trapezoidal rule approximation of
$$\int_0^1 e^{-x}\,dx$$
is accurate to within 10^{-5}?

22. How large should n be so that the trapezoidal rule approximation of
$$\int_0^2 \sin x\,dx$$
is accurate to within 10^{-4}?

23. How large should n be so that the trapezoidal rule approximation of
$$\int_1^2 \frac{e^t}{t}\,dt$$
is accurate to within 10^{-4}?

24. How large should n be so that the trapezoidal rule approximation of
$$\int_1^2 \frac{\cos x}{x}\,dx$$
is accurate to within 10^{-3}?

25. (a) Show graphically that, for $n = 5$, the trapezoidal rule overestimates, and the midpoint rule underestimates,
$$\int_0^1 x^3\,dx$$
In each case, compute the approximate value of the integral and compare it with the exact value.

(b) The result in (a) has to do with the fact that $y = x^3$ is concave up on $[0, 1]$. To generalize that result to functions with this concavity property, we assume that the function $f(x)$ is continuous, nonnegative, and concave up on the interval $[a, b]$. Denote by M_n the midpoint rule approximation, and by T_n the trapezoidal rule approximation, of $\int_a^b f(x)\,dx$. Explain in words why
$$M_n \le \int_a^b f(x)\,dx \le T_n$$

(c) If we assume that $f(x)$ is continuous, nonnegative, and concave *down* on $[a, b]$, then
$$M_n \ge \int_a^b f(x)\,dx \ge T_n$$
Explain why this is so. Use this result to give an upper and a lower bound on
$$\int_0^1 \sqrt{x}\,dx$$
when $n = 4$ in the approximation.

7.6 The Taylor Approximation

In many ways, polynomials are the easiest functions to work with. Therefore, in this section we will learn how to approximate functions by polynomials. We will see that the approximation typically improves when we use higher-degree polynomials.

7.6.1 Taylor Polynomials

In Section 4.8, we discussed how to linearize a function about a given point. This discussion led to the linear, or tangent, approximation. We found the following:

> The linear approximation of $f(x)$ at $x = a$ is
> $$L(x) = f(a) + f'(a)(x - a)$$

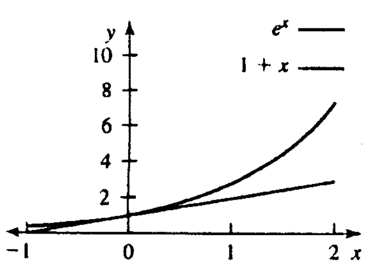

Figure 7.40 The graph of $y = e^x$ and its linear approximation at 0.

As an example, we look at
$$f(x) = e^x$$
and approximate this function by its linearization at $x = 0$. We find that
$$L(x) = f(0) + f'(0)x = 1 + x \qquad (7.19)$$
since $f'(x) = e^x$ and $f(0) = f'(0) = 1$. To see how close the approximation is, we graph both $f(x)$ and $L(x)$ in the same coordinate system. (The result is shown in Figure 7.40.) The approximation is quite good as long as x is close to 0. The figure suggests that it gets gradually worse as we move away from 0. In the approximation, we required only that $f(x)$ and $L(x)$ have in common $f(0) = L(0)$ and $f'(0) = L'(0)$.

a polynomial of degree 7 might not. We can easily check this; we find that

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{7!}=2.71825396825$$
$$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots+\frac{1}{8!}=2.71827876984$$

Comparing these with $e = 2.71828182846\ldots$, we see that the error is equal to 2.79×10^{-5} when $n = 7$ and 3.06×10^{-6} when $n = 8$. The error that we computed with (7.28) is a worst-case scenario; that is, the true error can be (and typically is) smaller than the error bound. ■

We have already seen one example in which a Taylor polynomial was useful only for values close to the point at which we approximated the function, regardless of n, the degree of the polynomial. In some situations, the error in the approximation cannot be made small for *any* value close to the point of approximation, regardless of n. One such example is the continuous function

$$f(x)=\begin{cases} e^{-1/x} & \text{for } x>0 \\ 0 & \text{for } x\le 0 \end{cases}$$

which is used, for instance, to describe the height of a tree as a function of age. We can show that $f^{(k)}(0) = 0$ for *all* $k \ge 1$, which implies that a Taylor polynomial of degree n about $x = 0$ is

$$P_n(x)=0$$

for all n. This example clearly shows that it will not help to increase n; the approximation just will not improve.

When we use Taylor polynomials to approximate functions, it is important to know for which values of x the approximation can be made arbitrarily close by choosing n large.

Following are a few of the most important functions, together with their Taylor polynomials about $x = 0$ and the range of x values over which the approximation can be made arbitrarily close by choosing n large enough:

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+R_{n+1}(x),\quad -\infty<x<\infty$$
$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+R_{n+1}(x),\quad -\infty<x<\infty$$
$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+R_{n+1}(x),\quad -\infty<x<\infty$$
$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots+(-1)^{n+1}\frac{x^n}{n}+R_{n+1}(x),\quad -1<x\le 1$$
$$\frac{1}{1-x} = 1+x+x^2+x^3+x^4+\cdots+x^n+R_{n+1}(x),\quad -1<x<1$$

Section 7.6 Problems

7.6.1

In Problems 1–5, find the linear approximation of $f(x)$ at $x = 0$.

1. $f(x) = e^{2x}$
2. $f(x) = \sin(3x)$
3. $f(x) = \dfrac{1}{1-x}$
4. $f(x) = x^4$
5. $f(x) = \ln(2+x^2)$

In Problems 6–10, compute the Taylor polynomial of degree n about $a = 0$ for the indicated functions.

6. $f(x) = \dfrac{1}{1+x}, n = 4$
7. $f(x) = \cos x, n = 5$
8. $f(x) = e^{3x}, n = 3$
9. $f(x) = x^5, n = 6$
10. $f(x) = \sqrt{1+x}, n = 3$

In Problems 11–16, compute the Taylor polynomial of degree n about a = 0 for the indicated functions and compare the value of the functions at the indicated point with the value of the corresponding Taylor polynomial.

11. $f(x) = \sqrt{2+x}, n = 3, x = 0.1$
12. $f(x) = \frac{1}{1-x}, n = 3, x = 0.1$
13. $f(x) = \sin x, n = 5, x = 1$
14. $f(x) = e^{-x}, n = 4, x = 0.3$
15. $f(x) = \tan x, n = 2, x = 0.1$
16. $f(x) = \ln(1+x), n = 3, x = 0.1$
17. **(a)** Find the Taylor polynomial of degree 3 about $a = 0$ for $f(x) = \sin x$.
(b) Use your result in (a) to give an intuitive explanation why
$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$
18. **(a)** Find the Taylor polynomial of degree 2 about $a = 0$ for $f(x) = \cos x$.
(b) Use your result in (a) to give an intuitive explanation why
$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

■ 7.6.2

In Problems 19–23, compute the Taylor polynomial of degree n about a and compare the value of the approximation with the value of the function at the given point x.

19. $f(x) = \sqrt{x}, a = 1, n = 3; x = 2$
20. $f(x) = \ln x, a = 1, n = 3; x = 2$
21. $f(x) = \cos x, a = \frac{\pi}{6}, n = 3; x = \frac{\pi}{7}$
22. $f(x) = x^{1/5}, a = -1, n = 3; x = -0.9$
23. $f(x) = e^x, a = 2, n = 3; x = 2.1$
24. Show that
$$T^4 \approx T_a^4 + 4T_a^3(T - T_a)$$
for T close to T_a.
25. Show that, for positive constants r and k,
$$rN\left(1 - \frac{N}{K}\right) \approx rN$$
for N close to 0.
26. **(a)** Show that, for positive constants a and k,
$$f(R) = \frac{aR}{k+R} \approx \frac{a}{k}R$$
for R close to 0.
(b) Show that, for positive constants a and k,
$$f(R) = \frac{aR}{k+R} \approx \frac{a}{2} + \frac{a}{4k}(R-k)$$
for R close to k.

■ 7.6.3

In Problems 27–30, use the following form of the error term
$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$
where c is between 0 and x, to determine in advance the degree of Taylor polynomial at a = 0 that would achieve the indicated accuracy in the interval [0, x]. (Do not compute the Taylor polynomial.)

27. $f(x) = e^x, x = 2$, error $< 10^{-3}$
28. $f(x) = \cos x, x = 1$, error $< 10^{-2}$
29. $f(x) = 1/(1+x), x = 0.2$, error $< 10^{-2}$
30. $f(x) = \ln(1+x), x = 0.1$, error $< 10^{-2}$
31. Let $f(x) = e^{-1/x}$ for $x > 0$ and $f(x) = 0$ for $x = 0$. Compute a Taylor polynomial of degree 2 at $x = 0$, and determine how large the error is.
32. We can show that the Taylor polynomial for $f(x) = (1+x)^\alpha$ about $x = 0$, with α a positive constant, converges for $x \in (-1, 1)$. Show that
$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + R_{n+1}(x)$$
33. We can show that the Taylor polynomial for $f(x) = \tan^{-1} x$ about $x = 0$ converges for $|x| \le 1$.
(a) Show that the following is true:
$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + R_{n+1}(x)$$
(b) Explain why the following holds:
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$
(This series converges very slowly, as you would see if you used it to approximate π.)

■ 7.7 Tables of Integrals (Optional)

Before the advent of software that could integrate functions, tables of indefinite integrals were useful aids for evaluating integrals. In using a table of integrals, it is still necessary to bring the integrand of interest into a form that is listed in the table—and there are many integrals that simply cannot be evaluated exactly and must be evaluated numerically. We will give a very brief list of indefinite integrals and explain how to use such tables.

■ 7.7.1 A Note on Software Packages That Can Integrate

Mathematicians and scientists use software packages to integrate functions. They are not difficult to use with some practice. Although they will not give you insight into what technique could be used to solve an integration problem, they quickly give you the correct answer. For instance, if we used MATLAB, one of the common software packages, to calculate the integral in Example 8 of Section 7.3, we would enter the following string of commands into our computer:

```
syms x;
f=x^4*(1-x)^4/(1+x^2);
int(f,0,1)
```

MATLAB then returns

```
ans = 22/7-pi
```

Section 7.7 Problems

In Problems 1–8, use the table on pages 383–384 to compute each integral.

1. $\int \frac{x}{2x-3}\,dx$
2. $\int \frac{dx}{16+x^2}$
3. $\int \sqrt{x^2-16}\,dx$
4. $\int \sin(2x)\cos(2x)\,dx$
5. $\int_0^1 x^3 e^{-x}\,dx$
6. $\int_0^{\pi/4} e^{-x}\cos(2x)\,dx$
7. $\int_1^e x^2 \ln x\,dx$
8. $\int_e^{e^2} \frac{dx}{x\ln x}$

In Problems 9–22, use the table on pages 383–384 to compute each integral after manipulating the integrand in a suitable way.

9. $\int_0^{\pi/6} e^x \cos\left(x-\frac{\pi}{6}\right)dx$
10. $\int_1^2 x\ln(2x-1)\,dx$
11. $\int (x^2-1)e^{-x/2}\,dx$
12. $\int (x+1)^2 e^{-2x}\,dx$
13. $\int \cos^2(5x-3)\,dx$
14. $\int \frac{x^2}{4x^2+4x+1}\,dx$
15. $\int \sqrt{9+4x^2}\,dx$
16. $\int \frac{1}{\sqrt{16-9x^2}}\,dx$
17. $\int e^{2x+1}\sin\left(\frac{\pi}{2}x\right)dx$
18. $\int (x-1)^2 e^{2x}\,dx$
19. $\int_2^4 \frac{1}{x\ln\sqrt{x}}\,dx$
20. $\int_1^e (x+2)^2\ln x\,dx$
21. $\int \cos(\ln(3x))\,dx$
22. $\int \frac{3}{x^2-4x+8}\,dx$

Chapter 7 Key Terms

Discuss the following definitions and concepts:

1. The substitution rule for indefinite integrals
2. The substitution rule for definite integrals
3. Integration by parts
4. The "trick" of "multiplying by 1"
5. Partial-fraction decomposition
6. Partial-fraction method
7. Proper rational function
8. Irreducible quadratic factor
9. Improper integral
10. Integration when the interval is unbounded
11. Integration when the integrand is discontinuous
12. Convergence and divergence of improper integrals
13. Comparison results for improper integrals
14. Numerical integration: midpoint and trapezoidal rule
15. Error bounds for the midpoint and the trapezoidal rule
16. Using tables of integrals for integration
17. Linear approximation
18. Taylor polynomial of degree n
19. Taylor's formula

Chapter 7 Review Problems

In Problems 1–30, evaluate the given indefinite integrals.

1. $\int x^2(1-x^3)^2\,dx$
2. $\int \frac{\cos x}{1+\sin^2 x}\,dx$
3. $\int 4xe^{-x^2}\,dx$
4. $\int \frac{x\ln(1+x^2)}{1+x^2}\,dx$
5. $\int (1+\sqrt{x})^{1/3}\,dx$
6. $\int x\sqrt{3x+1}\,dx$
7. $\int x\sec^2(3x^2)\,dx$
8. $\int \tan x\sec^2 x\,dx$
9. $\int x\ln x\,dx$
10. $\int x^3\ln x^2\,dx$

11. $\int \sec^2 x \ln(\tan x)\, dx$

12. $\int \sqrt{x} \ln \sqrt{x}\, dx$

13. $\int \frac{1}{4+x^2}\, dx$

14. $\int \frac{1}{4-x^2}\, dx$

15. $\int \tan x\, dx$

16. $\int \tan^{-1} x\, dx$

17. $\int e^{2x} \sin x\, dx$

18. $\int x \sin x\, dx$

19. $\int \sqrt{e^x}\, dx$

20. $\int \ln \sqrt{x}\, dx$

21. $\int \sin^2 x\, dx$

22. $\int \sin x \cos x\, e^{\sin x}\, dx$

23. $\int \frac{1}{x(x-1)}\, dx$

24. $\int \frac{1}{(x+1)(x-2)}\, dx$

25. $\int \frac{x}{x+5}\, dx$

26. $\int \frac{x}{x^2+5}\, dx$

27. $\int \frac{1}{x+5}\, dx$

28. $\int \frac{1}{x^2+5}\, dx$

29. $\int \frac{(x+1)^2}{x-1}\, dx$

30. $\int \frac{2x+1}{\sqrt{1-x^2}}\, dx$

In Problems 31–50, evaluate the given definite integrals.

31. $\int_1^3 \frac{x^2+1}{x}\, dx$

32. $\int_0^{\pi/2} x \sin x\, dx$

33. $\int_0^1 x e^{-x^2/2}\, dx$

34. $\int_1^2 \ln x\, dx$

35. $\int_0^2 \frac{1}{4+x^2}\, dx$

36. $\int_0^{1/2} \frac{2}{\sqrt{1-x^2}}\, dx$

37. $\int_2^6 \frac{1}{\sqrt{x-2}}\, dx$

38. $\int_0^2 \frac{1}{x-2}\, dx$

39. $\int_0^\infty \frac{1}{9+x^2}\, dx$

40. $\int_0^\infty \frac{1}{x^2+3}\, dx$

41. $\int_0^\infty \frac{1}{x+3}\, dx$

42. $\int_0^\infty \frac{1}{(x+3)^2}\, dx$

43. $\int_0^1 \frac{1}{x^2}\, dx$

44. $\int_1^\infty \frac{1}{x^2}\, dx$

45. $\int_0^1 \frac{1}{\sqrt{x}}\, dx$

46. $\int_1^\infty \frac{1}{\sqrt{x}}\, dx$

47. $\int_0^1 x \ln x\, dx$

48. $\int_0^1 x 2^x\, dx$

49. $\int_0^{\pi/4} e^{\cos x} \sin x\, dx$

50. $\int_0^{\pi/4} x \sin(2x)\, dx$

In Problems 51–54, use (a) the midpoint rule and (b) the trapezoidal rule to approximate each integral with the specified value of n.

51. $\int_0^2 (x^2-1)\, dx, n = 4$

52. $\int_{-1}^1 (x^3-1)\, dx, n = 4$

53. $\int_0^1 e^{-x}\, dx, n = 5$

54. $\int_0^{\pi/4} \sin(4x)\, dx, n = 4$

In Problems 55–58, find the Taylor polynomial of degree n about x = a for each function.

55. $f(x) = \sin(2x), a = 0, n = 3$

56. $f(x) = e^{-x^2/2}, a = 0, n = 3$

57. $f(x) = \ln x, a = 1, n = 3$

58. $f(x) = \frac{1}{x-3}, a = 4, n = 4$

59. Cost of Gene Substitution *(Adapted from Roughgarden, 1996)* Suppose that an advantageous mutation arises in a population. Initially, the gene carrying this mutation is at a low frequency. As the gene spreads through the population, the average fitness of the population increases. We denote by $f_{avg}(t)$ the average fitness of the population at time t, by $f_{avg}(0)$ the average fitness of the population at time 0 (when the mutation arose), and by K the final value of the average fitness after the mutation has spread through the population. Haldane (1957) suggested measuring the *cost of evolution* (now known as the *cost of gene substitution*) by the cumulative difference between the current and the final fitness—that is, by

$$\int_0^\infty (K - f_{avg}(t))\, dt$$

In Figure 7.43, shade the region whose area is equal to the cost of gene substitution.

Figure 7.43 The cost of gene substitution. See Problem 59.