Section 9.1 Problems

■ 9.1.1

In Problems 1–4, solve each linear system of equations. In addition, for each system, graph the two lines corresponding to the two equations in a single coordinate system and use your graph to explain your solution.

1. $x - y = 1$
$x - 2y = -2$

2. $2x + 3y = 6$
$x - 4y = -4$

3. $x - 3y = 6$
$y = 3 + \frac{1}{3}x$

4. $2x + y = \frac{1}{3}$
$6x + 3y = 1$

5. Determine c such that

$$2x - 3y = 5$$
$$4x - 6y = c$$

has **(a)** infinitely many solutions and **(b)** no solutions. **(c)** Is it possible to choose a number for c so that the system has exactly one solution? Explain your answer.

6. (a) Determine the solution of

$$-2x + 3y = 5$$
$$ax - y = 1$$

in terms of a.

(b) For which values of a are there no solutions, exactly one solution, and infinitely many solutions?

7. Show that the solution of

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

is given by

$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{21}a_{12}}$$

and

$$x_2 = \frac{-a_{21}b_1 + a_{11}b_2}{a_{11}a_{22} - a_{21}a_{12}}$$

8. Assume that the system

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

has infinitely many solutions. Determine the number of solutions if you change

(a) a_{11} **(b)** b_1

In Problems 9–16, reduce the system of linear equations to upper triangular form and solve.

9. $2x - y = 3$
$x - 3y = 7$

10. $5x - 3y = 2$
$2x + 7y = 3$

11. $7x - y = 4$
$3x + 2y = 1$

12. $5x + 2y = 8$
$-x + 3y = 9$

13. $3x - y = 1$
$-3x + y = 4$

14. $2x + 3y = 5$
$-y = -2 + \frac{2}{3}x$

15. $x + 2y = 3$
$4y + 2x = 6$

16. $x - 2y = 2$
$4y - 2x = -4$

17. Zach wants to buy fish and plants for his aquarium. Each fish costs \$2.30; each plant costs \$1.70. He buys a total of 11 items and spends a total of \$21.70. Set up a system of linear equations that will allow you to determine how many fish and how many plants Zach bought, and solve the system.

18. Laboratory mice are fed with a mixture of two foods that contain two essential nutrients. Food 1 contains 3 units of nutrient A and 2 units of nutrient B per ounce; food 2 contains 4 units of nutrient A and 5 units of nutrient B per ounce.

(a) In what proportion should you mix the food if the mice require nutrients A and B in equal amounts?

(b) Assume now that the mice require nutrients A and B in the ratio 1:2. Is it possible to satisfy their dietary needs with the two foods available?

19. Show that if

$$a_{11}a_{22} - a_{21}a_{12} \neq 0$$

then the system

$$a_{11}x_1 + a_{12}x_2 = 0$$
$$a_{21}x_1 + a_{22}x_2 = 0$$

has exactly one solution, namely, $x_1 = 0$ and $x_2 = 0$.

■ 9.1.2

In Problems 20–24, solve each system of linear equations.

20. $2x - 3y + z = -1$
$x + y - 2z = -3$
$3x - 2y + z = 2$

21. $5x - y + 2z = 6$
$x + 2y - z = -1$
$3x + 2y - 2z = 1$

22. $x + 4y - 3z = -13$
$2x - 3y + 5z = 18$
$3x + y - 2z = 1$

23. $-2x + 4y - z = -1$
$x + 7y + 2z = -4$
$3x - 2y + 3z = -3$

24. $2x - y + 3z = 3$
$2x + y + 4z = 4$
$2x - 3y + 2z = 2$

In Problems 25–28, find the augmented matrix and use it to solve the system of linear equations.

25. $-x - 2y + 3z = -9$
$2x + y - z = 5$
$4x - 3y + 5z = -9$

26. $3x - 2y + z = 4$
$4x + y - 2z = -12$
$2x - 3y + z = 7$

27. $y + x = 3$
$z - y = -1$
$x + z = 2$

28. $2x - z = 1$
$y + 3z = x - 1$
$x + z = y - 3$

In Problems 29–34, determine whether each system is overdetermined or underdetermined; then solve each system.

29. $x - 2y + z = 3$
$2x - 3y + z = 8$

30. $x - y = 2$
$x + y + z = 3$

31. $2x - y = 3$
$x - y = 4$
$3x - y = 1$

32. $4y - 3z = 6$
$2y + z = 1$
$y + z = 0$

33. $2x - 7y + z = 2$
$x + y - 2z = 4$

34. $3x + y = 1$
$x - y = 0$
$4x = 1$

35. SplendidLawn sells three types of lawn fertilizer: SL 24–4–8, SL 21–7–12 and SL 17–0–0. The three numbers refer to the percentages of nitrogen, phosphate, and potassium, in that order, of the contents. (For instance, 100 g of SL 24–4–8 contains 24 g of nitrogen.) Suppose that each year your lawn requires 500 g of nitrogen, 100 g of phosphate, and 180 g of potassium per 1000 square feet. How much of each of the three types of fertilizer should you apply per 1000 square feet per year?

36. Three different species of insects are reared together in a laboratory cage. They are supplied with two different types of food each day. Each individual of species 1 consumes 3 units of food A and 5 units of food B, each individual of species 2 consumes 2 units of food A and 3 units of food B, and individual of species 3 consumes 1 unit of food A and 2 units of food B. Each day, 500 units of food A and 900 units of food B are supplied. How many individuals of each species can be reared together? Is there more than one solution? What happens if we add 550 units of a third type of food, called C, and each individual of species 1 consumes 2 units of food C, each individual of species 2 consumes 4 units of food C, and each individual of species 3 consumes 1 unit of food C?

9.2 Matrices

We introduced matrices in the previous section; in this section, we will learn various matrix operations.

9.2.1 Basic Matrix Operations

Recall that an $m \times n$ matrix A is a rectangular array of numbers with m rows and n columns. We write it as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [\, a_{ij} \,]_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

We will also use the shorthand notation $A = [a_{ij}]$ if the size of the matrix is clear. We list a few simple definitions that do not need much explanation.

> **Definition** Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices. Then
>
> $$A = B$$
>
> if and only if, for all $1 \le i \le m$ and $1 \le j \le n$,
>
> $$a_{ij} = b_{ij}$$

This definition says that we can compare matrices of the same size, and they are equal if all their corresponding entries are equal. The next definition shows how to add matrices.

> **Definition** Suppose that $A = [a_{ij}]$ and $B = [b_{ij}]$ are two $m \times n$ matrices. Then
>
> $$C = A + B$$
>
> is an $m \times n$ matrix with entries
>
> $$c_{ij} = a_{ij} + b_{ij} \quad \text{for } 1 \le i \le m, 1 \le j \le n$$

Note that addition is defined only for matrices of equal size. Matrix addition satisfies the following two properties:

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$

Section 9.2 Problems

■ 9.2.1, 9.2.2

In Problems 1–6, let

$$A=\begin{bmatrix}-1 & 2\\ 0 & -3\end{bmatrix},\quad B=\begin{bmatrix}0 & 1\\ 2 & 4\end{bmatrix},\quad C=\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}$$

1. Find $A - B + 2C$.
2. Find $-2A + 3B$.
3. Determine D so that $A + B = 2A - B + D$.
4. Show that $A + B = B + A$.
5. Show that $(A + B) + C = A + (B + C)$.
6. Show that $2(A + B) = 2A + 2B$.

In Problems 7–12, let

$$A=\begin{bmatrix}1 & 3 & -1\\ 2 & 4 & 1\\ 0 & -2 & 2\end{bmatrix},\quad B=\begin{bmatrix}5 & -1 & 4\\ 2 & 0 & 1\\ 1 & -3 & -3\end{bmatrix},$$

$$C=\begin{bmatrix}-2 & 0 & 4\\ 1 & -3 & 1\\ 0 & 0 & 2\end{bmatrix}$$

7. Find $2A + 3B - C$.
8. Find $3C - B + \frac{1}{2}A$.
9. Determine D so that $A + B + C + D = 0$.
10. Determine D so that $A + 4B = 2(A + B) + D$.
11. Show that $A + B = B + A$.
12. Show that $(A + B) + C = A + (B + C)$.
13. Show that if $A + B = C$, then $A = C - B$.
14. Find the transpose of

$$A=\begin{bmatrix}2\\ -3\\ 5\end{bmatrix}$$

15. Find the transpose of

$$A=\begin{bmatrix}-1 & 0 & 3\\ 2 & 1 & -4\end{bmatrix}$$

16. Suppose A is a 2×2 matrix. Find conditions on the entries of A such that

$$A + A' = 0$$

17. Suppose that A and B are $m \times n$ matrices. Show that

$$(A + B)' = A' + B'$$

18. Suppose that A is an $m \times n$ matrix. Show that

$$(A')' = A$$

19. Suppose that A is an $m \times n$ matrix and k is a real number. Show that

$$(kA)' = kA'$$

20. Suppose that A is an $m \times k$ matrix and B is a $k \times n$ matrix. Show that

$$(AB)' = B'A'$$

In Problems 21–26, let

$$A=\begin{bmatrix}-1 & 0\\ 1 & 2\end{bmatrix},\quad B=\begin{bmatrix}2 & 3\\ -1 & 1\end{bmatrix},\quad C=\begin{bmatrix}1 & 2\\ 0 & -1\end{bmatrix}$$

21. Compute the following:
(a) AB **(b)** BA
22. Compute ABC.
23. Show that $AC \neq CA$.
24. Show that $(AB)C = A(BC)$.
25. Show that $(A + B)C = AC + BC$.
26. Show that $A(B + C) = AB + AC$.
27. Suppose that A is a 3×4 matrix and B is a 4×2 matrix. What is the size of the product AB?
28. Suppose A is a 3×4 matrix and B is an $m \times n$ matrix. What are values of m and n such that the following products are defined?
(a) AB **(b)** BA
29. Suppose that A is a 4×3 matrix, B is a 1×3 matrix, C is a 3×1 matrix, and D is a 4×3 matrix. Which of the matrix multiplications that follow are defined? Whenever it is defined, state the size of the resulting matrix.
(a) BD' **(b)** $D'A$ **(c)** ACB
30. Suppose that A is an $l \times p$ matrix, B is an $m \times q$, matrix, and C is an $n \times r$ matrix. What can you say about l, m, n, p, q, and r if the products that follow are defined? State the size of the resulting matrix.
(a) ABC **(b)** $AB'C$ **(c)** BAC' **(d)** $A'CB'$
31. Let

$$A=\begin{bmatrix}1 & 3\\ 0 & -2\end{bmatrix} \text{ and } B=\begin{bmatrix}1 & 2 & 0 & -1\\ 2 & 1 & 3 & 0\end{bmatrix}$$

(a) Compute AB. **(b)** Compute $B'A$.

32. Let

$$A=\begin{bmatrix}1 & 4 & -2\end{bmatrix} \text{ and } B=\begin{bmatrix}-1\\ 2\\ 3\end{bmatrix}$$

(a) Compute AB. **(b)** Compute BA.

33. Let

$$A=\begin{bmatrix}2 & 1\\ -1 & -3\end{bmatrix}$$

Find A^2, A^3, and A^4.

34. Suppose that

$$A=\begin{bmatrix}1 & -1\\ 3 & 0\\ 5 & 2\end{bmatrix} \text{ and } B=\begin{bmatrix}2 & 4 & 1\\ 6 & 0 & 0\end{bmatrix}$$

Show that $(AB)' = B'A'$.

35. Let

$$B=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

(a) Find B^2, B^3, B^4, and B^5.
(b) What can you say about B^k when **(i)** k is even and **(ii)** k is odd?

36. Let

$$I_3=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Show that $I_3 = I_3^2 = I_3^3$.

37. Let

$$A=\begin{bmatrix}1 & 3\\ 0 & -2\end{bmatrix} \text{ and } I_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Show that $AI_2 = I_2A = A$.

38. Let

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \\ -1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that $AI_3 = I_3A = A$.

In Problems 39–42, write each system in matrix form.

39. $$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 0 \\ 2x_2 + x_3 &= 1 \\ x_1 \quad - 2x_3 &= 2 \end{aligned}$$

40. $$\begin{aligned} 2x_2 - x_1 &= x_3 \\ 4x_1 + x_3 &= 7x_2 \\ x_2 - x_1 &= x_3 \end{aligned}$$

41. $$\begin{aligned} 2x_1 - 3x_2 &= 4 \\ -x_1 + x_2 &= 3 \\ 3x_1 &= 4 \end{aligned}$$

42. $$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

9.2.3

43. Show that the inverse of

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

is

$$B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

44. Show that the inverse of

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

is

$$B = \begin{bmatrix} -\frac{6}{5} & \frac{2}{5} & \frac{7}{5} \\ \frac{3}{5} & -\frac{1}{5} & -\frac{1}{5} \\ \frac{8}{5} & -\frac{1}{5} & -\frac{11}{5} \end{bmatrix}$$

In Problems 45–48, let

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$$

45. Find the inverse (if it exists) of A.

46. Find the inverse (if it exists) of B.

47. Show that $(A^{-1})^{-1} = A$.

48. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

49. Find the inverse (if it exists) of

$$C = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

50. Find the inverse (if it exists) of

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

51. Suppose that

$$A = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$$

Find X such that $AX = D$ by
(a) solving the associated system of linear equations and
(b) using the inverse of A.

52. **(a)** Show that if $X = AX + D$, then

$$X = (I - A)^{-1}D$$

provided that $I - A$ is invertible.
(b) Suppose that

$$A = \begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Compute $(I - A)^{-1}$, and use your result in (a) to compute X.

53. Use the determinant to determine whether the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

is invertible.

54. Use the determinant to determine whether the matrix

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 3 \end{bmatrix}$$

is invertible.

55. Use the determinant to determine whether the matrix

$$A = \begin{bmatrix} 4 & -1 \\ 8 & -2 \end{bmatrix}$$

is invertible.

56. Use the determinant to determine whether the matrix

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

is invertible.

57. Suppose that

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$$

(a) Compute $\det A$. Is A invertible?
(b) Suppose that

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Write $AX = B$ as a system of linear equations.
(c) Show that if

$$B = \begin{bmatrix} 3 \\ \frac{9}{2} \end{bmatrix}$$

then

$$AX = B$$

has infinitely many solutions. Graph the two straight lines associated with the corresponding system of linear equations, and explain why the system has infinitely many solutions.
(d) Find a column vector

$$B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

so that

$$AX = B$$

has no solutions.

58. Suppose that

$$A = \begin{bmatrix} a & 8 \\ 2 & 4 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

(a) Show that when $a \neq 4$, $AX = B$ has exactly one solution.

(b) Suppose $a = 4$. Find conditions on b_1 and b_2 such that $AX = B$ has **(i)** infinitely many solutions and **(ii)** no solutions.

(c) Explain your results in (a) and (b) graphically.

In Problems 59–62, use the determinant to find the inverse of A.

59. $A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$ **60.** $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

61. $A = \begin{bmatrix} -1 & 4 \\ 5 & 1 \end{bmatrix}$ **62.** $A = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

63. Use the determinant to determine whether

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

is invertible. If it is invertible, compute its inverse. In either case, solve $AX = \mathbf{0}$.

64. Use the determinant to determine whether

$$B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

is invertible. If it is invertible, compute its inverse. In either case, solve $BX = \mathbf{0}$.

65. Use the determinant to determine whether

$$C = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$$

is invertible. If it is invertible, compute its inverse. In either case, solve $CX = \mathbf{0}$.

66. Use the determinant to determine whether

$$D = \begin{bmatrix} -3 & 6 \\ -4 & 8 \end{bmatrix}$$

is invertible. If it is invertible, compute its inverse. In either case, solve $DX = \mathbf{0}$.

■ 9.2.4

In Problems 67–70, find the inverse matrix to each given matrix if the inverse matrix exists.

67. $A = \begin{bmatrix} 2 & -1 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ **68.** $A = \begin{bmatrix} -1 & 3 & -1 \\ 2 & -2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

69. $A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$ **70.** $A = \begin{bmatrix} -1 & 0 & 2 \\ -1 & -2 & 3 \\ 0 & 2 & -1 \end{bmatrix}$

■ 9.2.5

In Problems 71–74, suppose that breeding occurs once a year and that a census is taken at the end of each breeding season.

71. Assume that a population is divided into three age classes and that 20% of the females age 0 and 70% of the females age 1 survive until the end of the next breeding season. Assume further that females age 1 have an average of 3.2 female offspring and females age 2 have an average of 1.7 female offspring. If, at time 0, the population consists of 2000 females age 0, 800 females age 1, and 200 females age 2, find the Leslie matrix and the age distribution at time 2.

72. Assume that a population is divided into three age classes and that 80% of the females age 0 and 10% of the females age 1 survive until the end of the next breeding season. Assume further that females age 1 have an average of 1.6 female offspring and females age 2 have an average of 3.9 female offspring. If, at time 0, the population consists of 1000 females age 0, 100 females age 1, and 20 females age 2, find the Leslie matrix and the age distribution at time 3.

73. Assume that a population is divided into four age classes and that 70% of the females age 0, 50% of the females age 1, and 10% of the females age 2 survive until the end of the next breeding season. Assume further that females age 2 have an average of 4.6 female offspring and females age 3 have an average of 3.7 female offspring. If, at time 0, the population consists of 1500 females age 0, 500 females age 1, 250 females age 2, and 50 females age 3, find the Leslie matrix and the age distribution at time 2.

74. Assume that a population is divided into four age classes and that 65% of the females age 0, 40% of the females age 1, and 30% of the females age 2 survive until the end of the next breeding season. Assume further that females age 1 have an average of 2.8 female offspring, females age 2 have an average of 7.2 female offspring, and females age 3 have an average of 3.7 female offspring. If, at time 0, the population consists of 1500 females age 0, 500 females age 1, 250 females age 2, and 50 females age 3, find the Leslie matrix and the age distribution at time 3.

In Problems 75–76, assume the given Leslie matrix L. Determine the number of age classes in the population, the fraction of one-year-olds that survive until the end of the next breeding season, and the average number of female offspring of a two-year-old female.

75. $L = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$ **76.** $L = \begin{bmatrix} 0 & 5 & 0 \\ 0.8 & 0 & 0 \\ 0 & 0.3 & 0 \end{bmatrix}$

In Problems 77–78, assume the given Leslie matrix L. Determine the number of age classes in the population. What fraction of two-year-olds survive until the end of the next breeding season? Determine the average number of female offspring of a one-year-old female.

77. $L = \begin{bmatrix} 1 & 2.5 & 3 & 1.5 \\ 0.9 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{bmatrix}$ **78.** $L = \begin{bmatrix} 0 & 4.2 & 3.7 \\ 0.7 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix}$

79. Assume that the Leslie matrix is

$$L = \begin{bmatrix} 1.2 & 3.2 \\ 0.8 & 0 \end{bmatrix}$$

Suppose that, at time $t = 0$, $N_0(0) = 100$ and $N_1(0) = 0$. Find the population vectors for $t = 0, 1, 2, \ldots, 10$. Compute the successive ratios

$$q_0(t) = \frac{N_0(t)}{N_0(t-1)} \quad \text{and} \quad q_1(t) = \frac{N_1(t)}{N_1(t-1)}$$

for $t = 1, 2, \ldots, 10$. What value do $q_0(t)$ and $q_1(t)$ approach as $t \to \infty$? (Take a guess.) Compute the fraction of females age 0 for $t = 0, 1, \ldots, 10$. Can you find a stable age distribution?

80. Assume that the Leslie matrix is

$$L = \begin{bmatrix} 0.2 & 3 \\ 0.33 & 0 \end{bmatrix}$$

Suppose that, at time $t = 0$, $N_0(0) = 10$ and $N_1(0) = 5$. Find the population vectors for $t = 0, 1, 2, \ldots, 10$. Compute the successive ratios

$$q_0(t) = \frac{N_0(t)}{N_0(t-1)} \quad \text{and} \quad q_1(t) = \frac{N_1(t)}{N_1(t-1)}$$

for $t = 1, 2, \ldots, 10$. What value do $q_0(t)$ and $q_1(t)$ approach as $t \to \infty$? (Take a guess.) Compute the fraction of females age 0 for $t = 0, 1, \ldots, 10$. Can you find a stable age distribution?

81. Assume that the Leslie matrix is

$$L = \begin{bmatrix} 0 & 2 \\ 0.6 & 0 \end{bmatrix}$$

Suppose that, at time $t = 0$, $N_0(0) = 5$ and $N_1(0) = 1$. Find the population vectors for $t = 0, 1, 2, \ldots, 10$. Compute the successive ratios

$$q_0(t) = \frac{N_0(t)}{N_0(t-1)} \quad \text{and} \quad q_1(t) = \frac{N_1(t)}{N_1(t-1)}$$

for $t = 1, 2, \ldots, 10$. Do $q_0(t)$ and $q_1(t)$ converge? Compute the fraction of females age 0 for $t = 0, 1, \ldots, 10$. Describe the long-term behavior of $q_0(t)$.

82. Assume that the Leslie matrix is

$$L = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}$$

Suppose that, at time $t = 0$, $N_0(0) = 1$ and $N_1(0) = 1$. Find the population vectors for $t = 0, 1, 2, \ldots, 10$. Compute the successive ratios

$$q_0(t) = \frac{N_0(t)}{N_0(t-1)} \quad \text{and} \quad q_1(t) = \frac{N_1(t)}{N_1(t-1)}$$

for $t = 1, 2, \ldots, 10$. Do $q_0(t)$ and $q_1(t)$ converge? Compute the fraction of females age 0 for $t = 0, 1, \ldots, 10$. Describe the long-term behavior of $q_0(t)$.

9.3 Linear Maps, Eigenvectors, and Eigenvalues

In this section, we will denote vectors by boldface lowercase letters. Consider a map of the form

$$\mathbf{x} \to A\mathbf{x} \tag{9.18}$$

where A is a 2×2 matrix and $\mathbf{x}$ is a 2×1 column vector (or, simply, vector). Since $A\mathbf{x}$ is a 2×1 vector, this map takes a 2×1 vector and maps it into a 2×1 vector. That enables us to apply A repeatedly: We can compute $A(A\mathbf{x}) = A^2\mathbf{x}$, which is again a 2×1 vector, and so on. We will first look at vectors, then at maps $A\mathbf{x}$, and finally at iterates of the map A (i.e., $A^2\mathbf{x}$, $A^3\mathbf{x}$, and so on).

According to the properties of matrix multiplication, the map (9.18) satisfies the following conditions:

1. $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$, and
2. $A(\lambda\mathbf{x}) = \lambda(A\mathbf{x})$, where λ is a scalar.

Because of these two properties, we say that the map $\mathbf{x} \to A\mathbf{x}$ is *linear*.

We saw an example of such a map in the previous section: If A is a 2×2 Leslie matrix and $\mathbf{x}$ is a population vector at time 0, then $A\mathbf{x}$ represents the population vector at time 1.

Linear maps are important in other contexts as well, and we will encounter them in Chapters 10 and 11. Here, we restrict our discussion to 2×2 matrices but point out that we can generalize the discussion that follows to arbitrary $n \times n$ matrices. (These topics are covered in courses on matrix or linear algebra.)

9.3.1 Graphical Representation

Vectors We begin with a graphical representation of vectors. We assume that $\mathbf{x}$ is a 2×1 matrix. We call $\mathbf{x}$ a *column vector* or simply a *vector*. Since a 2×1 matrix has just two components, we can represent a vector in the plane. For instance, to represent the vector

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

in the x_1–x_2 plane, we draw an arrow from the origin $(0, 0)$ to the point $(3, 4)$, as illustrated in Figure 9.11. We see from Figure 9.11 that a vector has a length and a which is the same fraction of zero-year-olds at time 0. Furthermore, since $\lambda_1 > 0$, we can choose $\mathbf{u}_1$ so that both entries are positive (a condition that is needed if the entries represent population sizes). We summarize this demonstration as follows:

> If L is a 2×2 Leslie matrix with eigenvalues λ_1 and λ_2, then the eigenvector corresponding to the larger eigenvalue is a stable age distribution.

For the matrix

$$L = \begin{bmatrix} 1.5 & 2 \\ 0.08 & 0 \end{bmatrix}$$

the vector

$$\begin{bmatrix} 20 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to the larger eigenvalue; thus, it is a stable age distribution. In Section 9.2.5, we claimed that $\begin{bmatrix} 2000 \\ 100 \end{bmatrix}$ was a stable age distribution. In both cases, the fraction of zero-year-olds is the same, namely, $20/21 = 2000/2100$. That is, both vectors represent the same proportion of zero-year-olds in the population. We can check that $\begin{bmatrix} 2000 \\ 100 \end{bmatrix} = 100\begin{bmatrix} 20 \\ 1 \end{bmatrix}$, which shows that both vectors are eigenvectors. (Recall that if $\mathbf{u}$ is an eigenvector, then any vector $a\mathbf{u}$, $a \neq 0$, is also an eigenvector.) When we list a stable age distribution, we make sure that both entries are positive since they represent numbers of individuals in each age class.

If $\lambda_1 > |\lambda_2|$, then the population vector $N(t)$ will converge to a stable age distribution as $t \to \infty$, provided that $N(0) \neq \mathbf{u}_2$. This follows from writing $N(0)$ as a linear combination of the two eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ and applying L^t to the result; that is,

$$\begin{aligned} L^t N(0) &= L^t(a_1\mathbf{u}_1 + a_2\mathbf{u}_2) = a_1\lambda_1^t\mathbf{u}_1 + a_2\lambda_2^t\mathbf{u}_2 \\ &= a_1\lambda_1^t\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + a_2\lambda_2^t\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \end{aligned}$$

where $\mathbf{u}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Here, $a_1 \neq 0$, since $N(0) \neq \mathbf{u}_2$. The fraction of zero-year-olds at time t is

$$\frac{a_1\lambda_1^t x_1 + a_2\lambda_2^t x_2}{a_1\lambda_1^t x_1 + a_2\lambda_2^t x_2 + a_1\lambda_1^t y_1 + a_2\lambda_2^t y_2} \to \frac{x_1}{x_1 + y_1}$$

as $t \to \infty$.

Section 9.3 Problems

9.3.1

1. Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(a) Show by direct calculation that $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.

(b) Show by direct calculation that $A(\lambda\mathbf{x}) = \lambda(A\mathbf{x})$.

2. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

(a) Show by direct calculation that $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$.

(b) Show by direct calculation that $A(\lambda\mathbf{x}) = \lambda(A\mathbf{x})$.

In Problems 3–8, represent each given vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in the x_1–x_2 plane, and determine its length and the angle that it forms with the positive x_1-axis (measured counterclockwise).

3. $\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **4.** $\mathbf{x} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ **5.** $\mathbf{x} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

6. $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ **7.** $\mathbf{x} = \begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$ **8.** $\mathbf{x} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$

In Problems 9–12, vectors are given in their polar coordinate representation (length r, and angle α measured counterclockwise from the positive x_1-axis). Find the representation of the vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

in Cartesian coordinates.

9. $r = 2, \alpha = 30°$ **10.** $r = 3, \alpha = 150°$

11. $r = 1, \alpha = 120°$ **12.** $r = 5, \alpha = 240°$

13. Suppose a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has length 3 and is 15° clockwise from the positive x_1-axis. Find x_1 and x_2.

14. Suppose a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has length 2 and is 140° clockwise from the positive x_1-axis. Find x_1 and x_2.

15. Suppose a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has length 5 and is 25° counterclockwise from the positive x_2-axis. Find x_1 and x_2.

16. Suppose a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has length 4 and is 70° counterclockwise from the negative x_2-axis. Find x_1 and x_2.

In Problems 17–22, find $\mathbf{x} + \mathbf{y}$ *for the given vectors* $\mathbf{x}$ *and* $\mathbf{y}$. *Represent* $\mathbf{x}$, $\mathbf{y}$, *and* $\mathbf{x} + \mathbf{y}$ *in the plane, and explain graphically how you add* $\mathbf{x}$ *and* $\mathbf{y}$.

17. $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ **18.** $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

19. $\mathbf{x} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **20.** $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

21. $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ **22.** $\mathbf{x} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

In Problems 23–28, compute $a\mathbf{x}$ *for the given vector* $\mathbf{x}$ *and scalar* a. *Represent* $\mathbf{x}$ *and* $a\mathbf{x}$ *in the plane, and explain graphically how you obtain* $a\mathbf{x}$.

23. $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $a = 2$ **24.** $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $a = -1$

25. $\mathbf{x} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$ and $a = 0.5$ **26.** $\mathbf{x} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$ and $a = -1/3$

27. $\mathbf{x} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ and $a = 1/4$ **28.** $\mathbf{x} = \begin{bmatrix} 0.5 \\ 0.25 \end{bmatrix}$ and $a = 5$

In Problems 29–34, let

$$\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \quad \textit{and} \quad \mathbf{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

29. Compute $\mathbf{u} + \mathbf{v}$ and illustrate the result graphically.

30. Compute $\mathbf{u} - \mathbf{v}$ and illustrate the result graphically.

31. Compute $\mathbf{w} - \mathbf{u}$ and illustrate the result graphically.

32. Compute $\mathbf{v} - \frac{1}{2}\mathbf{u}$ and illustrate the result graphically.

33. Compute $\mathbf{u} + \mathbf{v} + \mathbf{w}$ and illustrate the result graphically.

34. Compute $2\mathbf{v} - \mathbf{w}$ and illustrate the result graphically.

In Problems 35–40, give a geometric interpretation of the map $\mathbf{x} \to A\mathbf{x}$ *for each given map A.*

35. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **36.** $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

37. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ **38.** $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

39. $A = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ **40.** $A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$

41. Use a rotation matrix to rotate the vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ counterclockwise by the angle $\pi/6$.

42. Use a rotation matrix to rotate the vector $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ counterclockwise by the angle $\pi/3$.

43. Use a rotation matrix to rotate the vector $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ counterclockwise by the angle $\pi/12$.

44. Use a rotation matrix to rotate the vector $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$ counterclockwise by the angle $\pi/9$.

45. Use a rotation matrix to rotate the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ clockwise by the angle $\pi/4$.

46. Use a rotation matrix to rotate the vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ clockwise by the angle $\pi/3$.

47. Use a rotation matrix to rotate the vector $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ clockwise by the angle $\pi/7$.

48. Use a rotation matrix to rotate the vector $\begin{bmatrix} -2 \\ -3 \end{bmatrix}$ clockwise by the angle $\pi/8$.

9.3.2

In Problems 49–56, find the eigenvalues λ_1 *and* λ_2 *and corresponding eigenvectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *for each matrix A. Determine the equations of the lines through the origin in the direction of the eigenvectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$, *and graph the lines together with the eigenvectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *and the vectors* $A\mathbf{v}_1$ *and* $A\mathbf{v}_2$.

49. $A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}$ **50.** $A = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}$

51. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **52.** $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

53. $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$ **54.** $A = \begin{bmatrix} 3 & 6 \\ -1 & -4 \end{bmatrix}$

55. $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ **56.** $A = \begin{bmatrix} -3 & -0.5 \\ 7 & 1.5 \end{bmatrix}$

In Problems 57–60, find the eigenvalues λ_1 *and* λ_2 *for each matrix A.*

57. $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ **58.** $A = \begin{bmatrix} -7 & 0 \\ 0 & 6 \end{bmatrix}$

59. $A = \begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}$ **60.** $A = \begin{bmatrix} -1 & 4 \\ 0 & -2 \end{bmatrix}$

61. Find the eigenvalues λ_1 and λ_2 for

$$A = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$$

62. Find the eigenvalues λ_1 and λ_2 for

$$A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

63. Let

$$A = \begin{bmatrix} 2 & 4 \\ -2 & -3 \end{bmatrix}$$

Without explicitly computing the eigenvalues of A, decide whether the real parts of both eigenvalues are negative.

64. Let

$$A = \begin{bmatrix} -2 & 3 \\ -1 & 1 \end{bmatrix}$$

Without explicitly computing the eigenvalues of A, decide whether the real parts of both eigenvalues are negative.

65. Let

$$A = \begin{bmatrix} 4 & 4 \\ -4 & -3 \end{bmatrix}$$

Without explicitly computing the eigenvalues of A, decide whether the real parts of both eigenvalues are negative.

66. Let

$$A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

Without explicitly computing the eigenvalues of A, decide whether the real parts of both eigenvalues are negative.

67. Let

$$A = \begin{bmatrix} 2 & -5 \\ 2 & -3 \end{bmatrix}$$

Without explicitly computing the eigenvalues of A, decide whether the real parts of both eigenvalues are negative.

68. Let

$$A = \begin{bmatrix} -2 & 5 \\ 2 & 3 \end{bmatrix}$$

Without explicitly computing the eigenvalues of A, decide whether the real parts of both eigenvalues are negative.

9.3.3

69. Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

(a) Show that

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

are eigenvectors of A and that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

(b) Represent

$$\mathbf{x} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(c) Use your results in (a) and (b) to compute $A^{20}\mathbf{x}$.

70. Let

$$A = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}$$

(a) Show that

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are eigenvectors of A and that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

(b) Represent

$$\mathbf{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$.

(c) Use your results in (a) and (b) to compute $A^{10}\mathbf{x}$.

71. Let

$$A = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix}$$

Find

$$A^{15}\begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

without using a calculator.

72. Let

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$$

Find

$$A^{30}\begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

without using a calculator.

73. Let

$$A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$$

Find

$$A^{20}\begin{bmatrix} -3 \\ -2 \end{bmatrix}$$

without using a calculator.

74. Let

$$A = \begin{bmatrix} 1 & -1/4 \\ 1/2 & 1/4 \end{bmatrix}$$

Find

$$A^{30}\begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

without using a calculator.

75. Suppose that

$$L = \begin{bmatrix} 2 & 4 \\ 0.3 & 0 \end{bmatrix}$$

is the Leslie matrix for a population with two age classes.

(a) Determine both eigenvalues.

(b) Give a biological interpretation of the larger eigenvalue.

(c) Find the stable age distribution.

76. Suppose that

$$L = \begin{bmatrix} 1 & 3 \\ 0.7 & 0 \end{bmatrix}$$

is the Leslie matrix for a population with two age classes.

(a) Determine both eigenvalues.

(b) Give a biological interpretation of the larger eigenvalue.

(c) Find the stable age distribution.

77. Suppose that

$$L = \begin{bmatrix} 7 & 3 \\ 0.1 & 0 \end{bmatrix}$$

is the Leslie matrix for a population with two age classes.

(a) Determine both eigenvalues.

(b) Give a biological interpretation of the larger eigenvalue.

(c) Find the stable age distribution.

78. Suppose that

$$L = \begin{bmatrix} 0 & 5 \\ 0.9 & 0 \end{bmatrix}$$

is the Leslie matrix for a population with two age classes.

(a) Determine both eigenvalues.

(b) Give a biological interpretation of the larger eigenvalue.

(c) Find the stable age distribution.

79. Suppose that

$$L = \begin{bmatrix} 0 & 5 \\ 0.09 & 0 \end{bmatrix}$$

is the Leslie matrix for a population with two age classes.

(a) Determine both eigenvalues.

(b) Give a biological interpretation of the larger eigenvalue.

(c) Find the stable age distribution.

Eliminating t in the parametric equation for a line in $\mathbf{R}^3$ is not very useful, since doing so does not yield just one equation as in the case of a line in $\mathbf{R}^2$. We will therefore forgo eliminating t in $\mathbf{R}^3$.

Section 9.4 Problems

■ 9.4.1

1. Let $\mathbf{x} = [1, 4, -1]'$ and $\mathbf{y} = [-2, 1, 0]'$.
(a) Find $\mathbf{x} + \mathbf{y}$. (b) Find $2\mathbf{x}$. (c) Find $-3\mathbf{y}$.

2. Let $\mathbf{x} = [-4, 3, 1]'$ and $\mathbf{y} = [0, -2, 3]'$.
(a) Find $\mathbf{x} - \mathbf{y}$. (b) Find $2\mathbf{x} + 3\mathbf{y}$. (c) Find $-\mathbf{x} - 2\mathbf{y}$.

3. Let $A = (2, 3)$ and $B = (4, 1)$. Find the vector representation of $\overrightarrow{AB}$.

4. Let $A = (-1, 0)$ and $B = (2, -4)$. Find the vector representation of $\overrightarrow{AB}$.

5. Let $A = (0, 1, -3)$ and $B = (-1, -1, 2)$. Find the vector representation of $\overrightarrow{AB}$.

6. Let $A = (1, 3, -2)$ and $B = (0, -1, 0)$. Find the vector representation of $\overrightarrow{AB}$.

7. Find the length of $\mathbf{x} = [1, 3]'$.

8. Find the length of $\mathbf{x} = [-2, 7]'$.

9. Find the length of $\mathbf{x} = [0, 1, 5]'$.

10. Find the length of $\mathbf{x} = [-2, 1, -3]'$.

11. Normalize $[1, 3, -1]'$.

12. Normalize $[2, 0, -4]'$.

13. Normalize $[6, 0, 0]'$.

14. Normalize $[0, -3, 1, 3]'$.

■ 9.4.2

15. Find the dot product of $\mathbf{x} = [1, 2]'$ and $\mathbf{y} = [3, -1]'$.

16. Find the dot product of $\mathbf{x} = [-1, 2]'$ and $\mathbf{y} = [-3, -4]'$.

17. Find the dot product of $\mathbf{x} = [0, -1, 3]'$ and $\mathbf{y} = [-3, 1, 1]'$.

18. Find the dot product of $\mathbf{x} = [2, -3, 1]'$ and $\mathbf{y} = [3, 1, -2]'$.

19. Use the dot product to compute the length of $[0, -1, 2]'$.

20. Use the dot product to compute the length of $[-1, 4, 3]'$.

21. Use the dot product to compute the length of $[1, 2, 3, 4]'$.

22. Use the dot product to compute the length of $[-1, -2, -3, -4]'$.

23. Find the angle between $\mathbf{x} = [1, 2]'$ and $\mathbf{y} = [3, -1]'$.

24. Find the angle between $\mathbf{x} = [-1, 2]'$ and $\mathbf{y} = [-2, -4]'$.

25. Find the angle between $\mathbf{x} = [0, -1, 3]'$ and $\mathbf{y} = [-3, 1, 1]'$.

26. Find the angle between $\mathbf{x} = [1, -3, 2]'$ and $\mathbf{y} = [3, 1, -4]'$.

27. Let $\mathbf{x} = [1, -1]'$. Find $\mathbf{y}$ so that $\mathbf{x}$ and $\mathbf{y}$ are perpendicular.

28. Let $\mathbf{x} = [-2, 1]'$. Find $\mathbf{y}$ so that $\mathbf{x}$ and $\mathbf{y}$ are perpendicular.

29. Let $\mathbf{x} = [1, -2, 4]'$. Find $\mathbf{y}$ so that $\mathbf{x}$ and $\mathbf{y}$ are perpendicular.

30. Let $\mathbf{x} = [2, 0, -1]'$. Find $\mathbf{y}$ so that $\mathbf{x}$ and $\mathbf{y}$ are perpendicular.

31. A triangle has vertices at coordinates $P = (0, 0)$, $Q = (4, 0)$, and $R = (4, 3)$.
(a) Use basic trigonometry to compute the lengths of all three sides and the measures of all three angles.
(b) Use the results of this section to repeat (a).

32. A triangle has vertices at coordinates $P = (0, 0)$, $Q = (0, 3)$, and $R = (5, 0)$.
(a) Use basic trigonometry to compute the lengths of all three sides and the measures of all three angles.
(b) Use the results of this section to repeat (a).

33. A triangle has vertices at coordinates $P = (1, 2, 3)$, $Q = (1, 5, 2)$, and $R = (2, 4, 2)$.
(a) Compute the lengths of all three sides.
(b) Compute all three angles in both radians and degrees.

34. A triangle has vertices at coordinates $P = (2, 1, 5)$, $Q = (-1, -3, 7)$, and $R = (2, -4, 1)$.
(a) Compute the lengths of all three sides.
(b) Compute all three angles in both radians and degrees.

35. Find the equation of the line through (2, 1) and perpendicular to $[1, 2]'$.

36. Find the equation of the line through (3, 2) and perpendicular to $[-1, 1]'$.

37. Find the equation of the line through $(1, -2)$ and perpendicular to $[4, 1]'$.

38. Find the equation of the line through (0, 1) and perpendicular to $[1, 0]'$.

39. Find the equation of the plane through (1, 2, 3) and perpendicular to $[0, -1, 1]'$.

40. Find the equation of the plane through $(1, 0, -3)$ and perpendicular to $[1, -2, -1]'$.

41. Find the equation of the plane through (0, 0, 0) and perpendicular to $[1, 0, 0]'$.

42. Find the equation of the plane through $(3, -1, 2)$ and perpendicular to $[-1, 1, 2]'$.

■ 9.4.3

In Problems 43–46, find the parametric equation of the line in the x–y plane that goes through the indicated point in the direction of the indicated vector.

43. $(1, -1), \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

44. $(3, -4), \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

45. $(-1, -2), \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

46. $(-1, 4), \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

In Problems 47–50, find the parametric equation of the line in the x–y plane that goes through the given points. Then eliminate the parameter to find the equation of the line in standard form.

47. $(-1, 2)$ and $(3, 4)$

48. $(2, 1)$ and $(3, 5)$

49. $(1, -3)$ and $(4, 0)$

50. $(2, 3)$ and $(-1, -4)$

In Problems 51–54, parameterize the equation of the line given in standard form.

51. $3x + 4y - 1 = 0$

52. $x - 2y + 5 = 0$

53. $2x + y - 3 = 0$

54. $x - 5y + 7 = 0$

In Problems 55–58, find the parametric equation of the line in x–y–z space that goes through the indicated point in the direction of the indicated vector.

55. $(1, -1, 2), \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

56. $(2, 0, 4), \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

57. $(-1, 3, -2), \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$

58. $(2, 1, -3), \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

In Problems 59–62, find the parametric equation of the line in x-y-z space that goes through the given points.

59. (5, 4, −1) and (2, 0, 3) **60.** (2, 0, −3) and (4, 1, 0)

61. (2, −3, 1) and (−5, 2, 1) **62.** (1, 0, 4) and (3, 2, 0)

63. Given are (1) a plane through (1, −1, 2) and perpendicular to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and (2) a line through the points (0, −3, 2) and (−1, −2, 3). Where do the plane and the line intersect?

64. Given are (1) a plane through (2, 0, −1) and perpendicular to $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ and (2) a line through the points (1, 0, −2) and (−1, −1, 1). Where do the plane and the line intersect?

65. Given is a plane through (0, −2, 1) and perpendicular to $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$. Find a line through (5, −1, 0) and that is parallel to the plane.

66. Given is the plane $x+2y-z+1=0$. Find a line in parametric form that is perpendicular to the plane.

Chapter 9 Key Terms

Discuss the following definitions and concepts:

1. Linear system of equations
2. Solving a linear system of equations
3. Upper triangular form
4. Gaussian elimination
5. Matrix
6. Basic matrix operations; addition, multiplication by a scalar
7. Transposition
8. Matrix multiplication
9. Identity matrix
10. Inverse matrix
11. Determinant
12. Leslie matrix
13. Stable age distribution
14. Vector
15. Parallelogram law
16. Linear map
17. Eigenvalues and eigenvectors
18. Vector addition
19. Multiplication of a vector by a scalar
20. Length of a vector
21. Dot product
22. Angle between two vectors
23. Perpendicular vectors
24. Line in the plane and in space
25. Equation of a plane
26. Parametric equation of a line

Chapter 9 Review Problems

1. Let

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 2 \end{bmatrix}$$

(a) Find $A\mathbf{x}$ when $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Graph both $\mathbf{x}$ and $A\mathbf{x}$ in the same coordinate system.

(b) Find the eigenvalues λ_1 and λ_2, and the corresponding eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$, of A.

(c) If $\mathbf{u}_i$ is the eigenvector corresponding to λ_i, find $A\mathbf{u}_i$ and explain graphically what happens when you apply A to $\mathbf{u}_i$.

(d) Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$; that is, find a_1 and a_2 so that

$$\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$$

Show that

$$A\mathbf{x} = a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2$$

and illustrate this equation graphically.

2. Let

$$A = \begin{bmatrix} 3 & 1/2 \\ -5 & -1/2 \end{bmatrix}$$

(a) Find $A\mathbf{x}$ when $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graph both $\mathbf{x}$ and $A\mathbf{x}$ in the same coordinate system.

(b) Find the eigenvalues λ_1 and λ_2, and the corresponding eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$, of A.

(c) If $\mathbf{u}_i$ is the eigenvector corresponding to λ_i, find $A\mathbf{u}_i$, and explain graphically what happens when you apply A to $\mathbf{u}_i$.

(d) Write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$; that is, find a_1 and a_2 so that

$$\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2$$

Show that

$$A\mathbf{x} = a_1\lambda_1\mathbf{u}_1 + a_2\lambda_2\mathbf{u}_2$$

and illustrate this equation graphically.

3. Given the Leslie matrix

$$L = \begin{bmatrix} 1.5 & 0.875 \\ 0.5 & 0 \end{bmatrix}$$

find the growth rate of the population and determine its stable age distribution.

4. Given the Leslie matrix

$$L = \begin{bmatrix} 0.5 & 2.99 \\ 0.25 & 0 \end{bmatrix}$$

find the growth rate of the population and determine its stable age distribution.

5. Let

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 4 & -1 \\ 8 & -1 \end{bmatrix}$$

Find B.

6. Let

$$(AB)^{-1} = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$$

Find A.

7. Explain two different ways to solve a system of the form

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

when $a_{11}a_{22} - a_{12}a_{21} \neq 0$.

8. Suppose that

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

has infinitely many solutions. If you wrote this system in matrix form $AX = B$, could you find X by computing $A^{-1}B$?

9. Let

$$\begin{aligned} ax + 3y &= 0 \\ x - y &= 0 \end{aligned}$$

How do you need to choose a so that this system has infinitely many solutions?

10. Let A be a 2×2 matrix and X and B be 2×1 matrices. Assume that $\det A = 0$. Explain how the choice of B affects the number of solutions of $AX = B$.

11. Suppose that

$$L = \begin{bmatrix} 0.5 & 2.3 \\ a & 0 \end{bmatrix}$$

is the Leslie matrix of a population with two age classes. For which values of a does this population grow?

12. Suppose that

$$L = \begin{bmatrix} 0.5 & 2.0 \\ 0.1 & 0 \end{bmatrix}$$

is the Leslie matrix of a population with two age classes.

(a) If you were to manage this population, would you need to be concerned about its long-term survival?

(b) Suppose that you can improve either the fecundity or the survival of the zero-year-olds, but due to physiological and environmental constraints, the fecundity of zero-year-olds will not exceed 1.5 and the survival of zero-year-olds will not exceed 0.4. Investigate how the growth rate of the population is affected by changing either the survival or the fecundity of zero-year-olds, or both. What would be the maximum achievable growth rate?

(c) In real situations, what other factors might you need to consider when you decide on management strategies?

13. Show that the eigenvalues of

$$\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$$

are equal to a and b.

14. Show that the eigenvalues of

$$\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$$

are equal to a and b.