Section 9.3

1.) $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

a.) $A(X+Y) = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} X_1 + Y_1 \\ X_2 + Y_2 \end{bmatrix}$

$$= \begin{bmatrix} (2)(X_1+Y_1) + (1)(X_2+Y_2) \\ (3)(X_1+Y_1) + (4)(X_2+Y_2) \end{bmatrix}$$

$$= \begin{bmatrix} 2X_1 + 2Y_1 + X_2 + Y_2 \\ 3X_1 + 3Y_1 + 4X_2 + 4Y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2X_1 + X_2 \\ 3X_1 + 4X_2 \end{bmatrix} + \begin{bmatrix} 2Y_1 + Y_2 \\ 3Y_1 + 4Y_2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$= AX + BY$$

b.) $A(\lambda X) = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \cdot \lambda \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \lambda X_1 \\ \lambda X_2 \end{bmatrix}$

$$= \begin{bmatrix} 2\lambda X_1 + \lambda X_2 \\ 3\lambda X_1 + 4\lambda X_2 \end{bmatrix} = \lambda \begin{bmatrix} 2X_1 + X_2 \\ 3X_1 + 4X_2 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \lambda (AX)$$

2.) $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$

a.) $A(X+Y) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 + Y_1 \\ X_2 + Y_2 \end{bmatrix}$

$$= \begin{bmatrix} a_{11}(x_1+y_1) + a_{12}(x_2+y_2) \\ a_{21}(x_1+y_1) + a_{22}(x_2+y_2) \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{11}y_1 + a_{12}y_2 \\ a_{21}x_1 + a_{22}x_2 + a_{21}y_1 + a_{22}y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{11}y_1 + a_{12}y_2 \\ a_{21}y_1 + a_{22}y_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= AX + AY$$

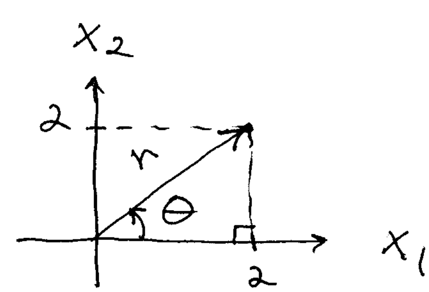
b.) $A(\lambda X) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} = \begin{bmatrix} a_{11}\lambda x_1 + a_{12}\lambda x_2 \\ a_{21}\lambda x_1 + a_{22}\lambda x_2 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \lambda \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \lambda (AX)$$

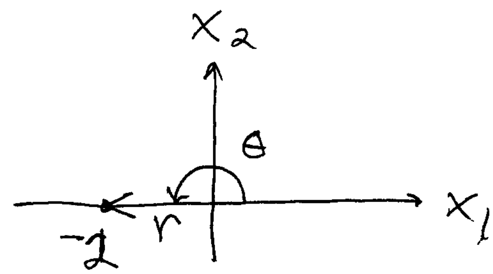
3.) $X = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$



$r = \sqrt{2^2 + 2^2} = \sqrt{8}$,

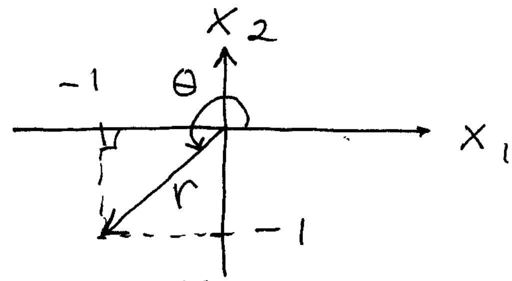
$\theta = \frac{\pi}{4}$

4.) $X = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$



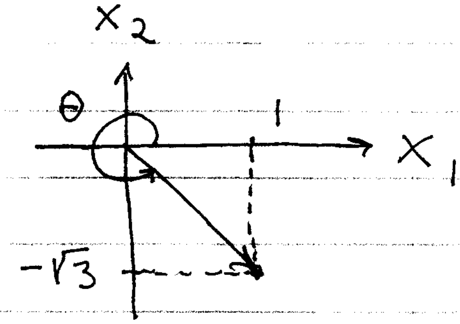
$r = 2, \qquad \theta = \pi$

6.) $X = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$



$r = \sqrt{1^2 + 1^2} = \sqrt{2}, \qquad \theta = \frac{5\pi}{4}$

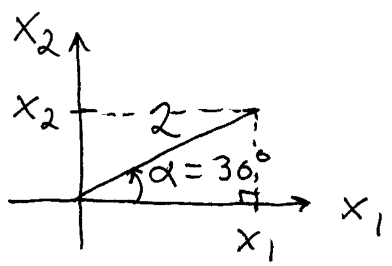
8.) $X = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$



$r = \sqrt{(\sqrt{3})^2 + 1^2} = 2,$

$\theta = \frac{5\pi}{3}$

9.)



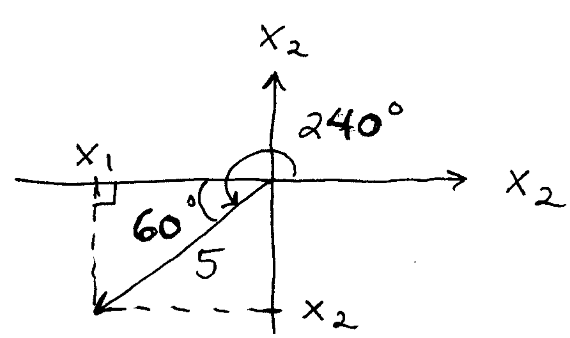
$\cos 30° = x_1/2 \rightarrow$

$x_1 = 2\cos 30° \rightarrow$

$x_1 = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3},$

$\sin 30° = \frac{x_2}{2} \rightarrow x_2 = 2\sin 30° = 2\left(\frac{1}{2}\right) = 1$

$\rightarrow X = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$



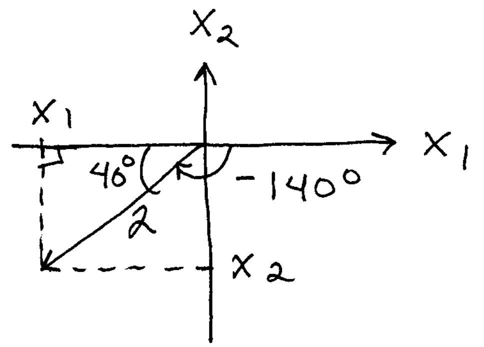
12.) $\sin 60° = \frac{|x_2|}{5}$

$\rightarrow |x_2| = 5\sin 60°$

$= 5\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2} \rightarrow x_2 = \frac{-5\sqrt{3}}{2};$

$\cos 60° = \frac{|x_1|}{5} \rightarrow |x_1| = 5\cos 60° = 5\left(\frac{1}{2}\right) = \frac{5}{2} \rightarrow$

$x_1 = \frac{-5}{2}$; then $X = \begin{bmatrix} -5/2 \\ -5\sqrt{3}/2 \end{bmatrix}$

14.) 

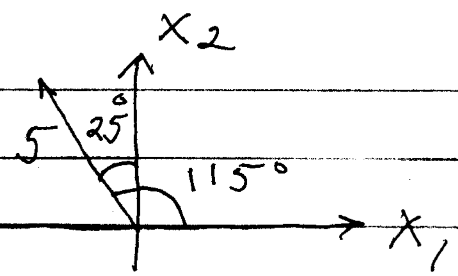
$\cos 40^\circ = \dfrac{|x_1|}{2} \rightarrow$

$|x_1| = 2\cos 40^\circ \rightarrow$

$x_1 = -2\cos 40^\circ$;

$\sin 40^\circ = \dfrac{|x_2|}{2} \rightarrow |x_2| = 2\sin 40^\circ \rightarrow$

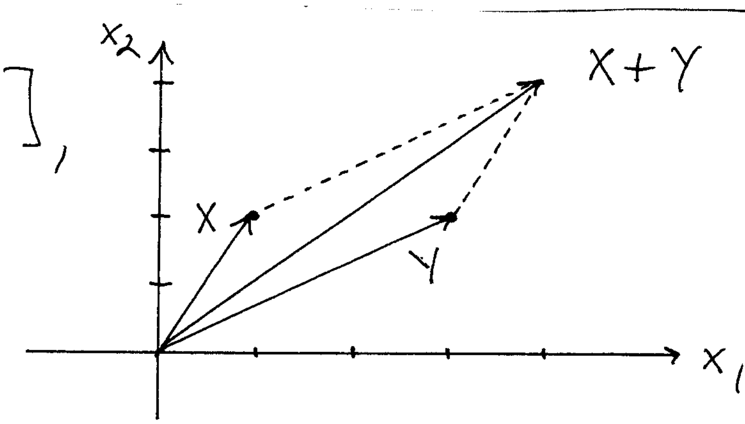
$x_2 = -2\sin 40^\circ \rightarrow X = \begin{bmatrix} -2\cos 40^\circ \\ -2\sin 40^\circ \end{bmatrix}$

15.) 

$X = \begin{bmatrix} 5\cos 115^\circ \\ 5\sin 115^\circ \end{bmatrix}$

17.) $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $Y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$,

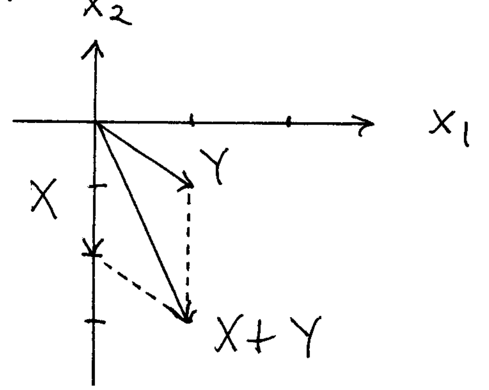
$X + Y = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$



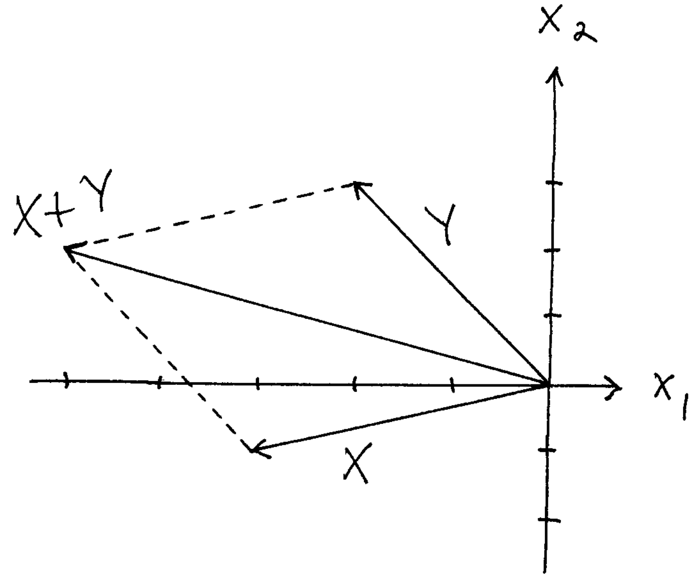
19.) $X = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$,

$Y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$,

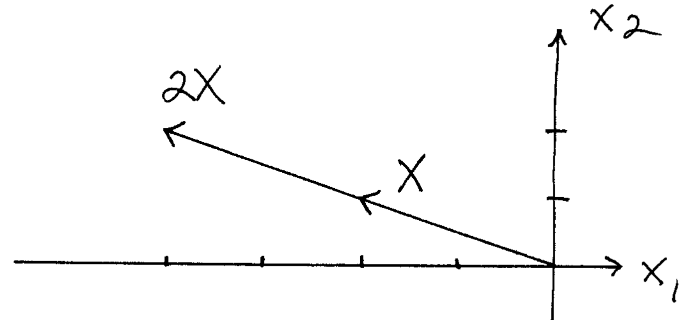
$X + Y = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$



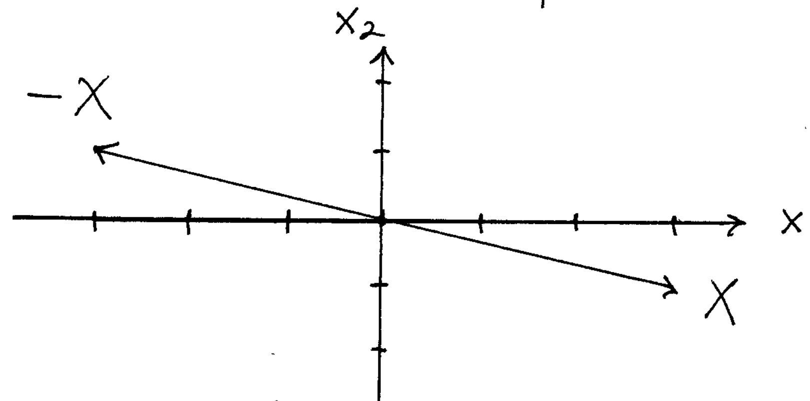
22.) $X = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, $Y = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $X+Y = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$



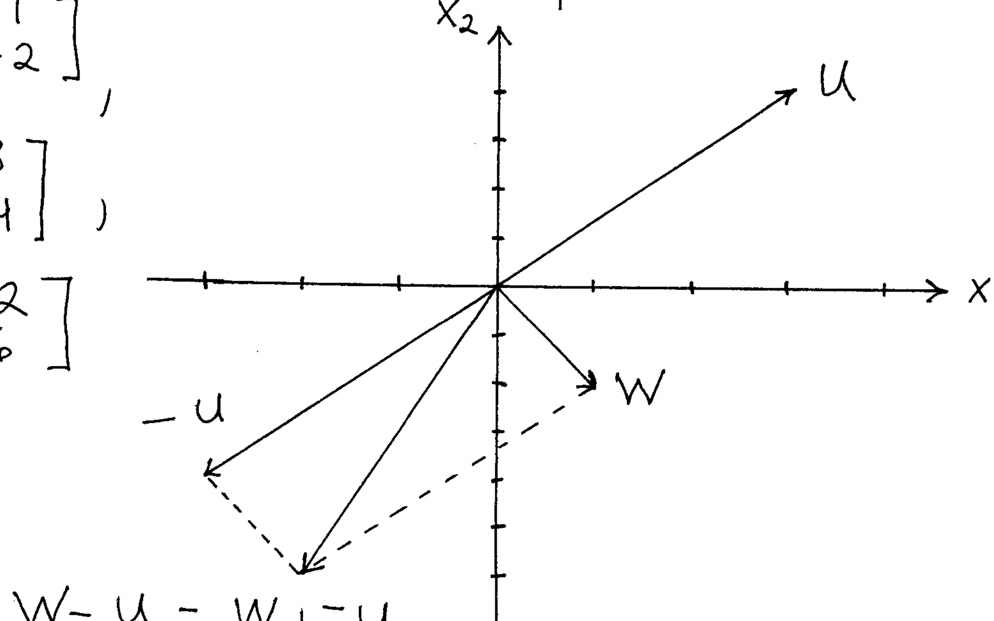
23.) $X = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $2X = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$



24.) $X = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $-X = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

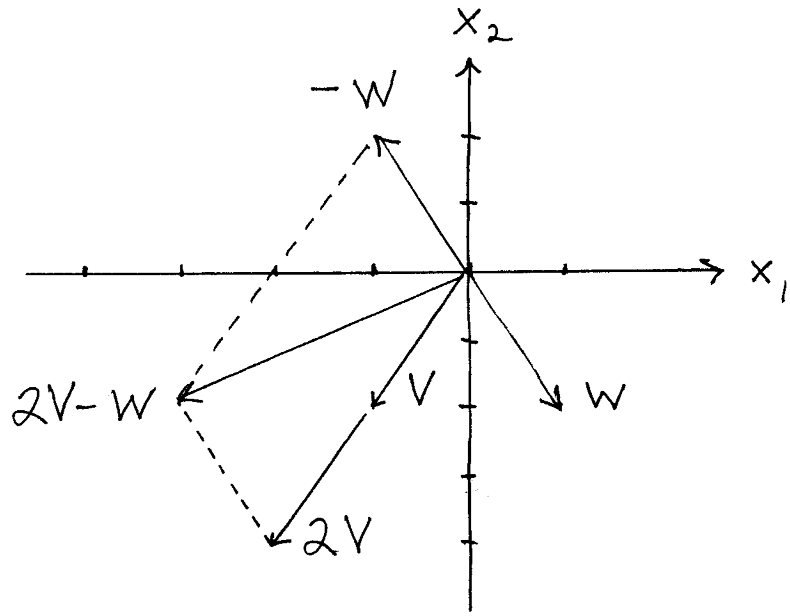


31.) $W = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $u = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $W - U = \begin{bmatrix} -2 \\ -6 \end{bmatrix}$



34.) $V = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $W = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$,

$2V - W = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$



37.) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos \pi/2 & -\sin \pi/2 \\ \sin \pi/2 & \cos \pi/2 \end{bmatrix}$,

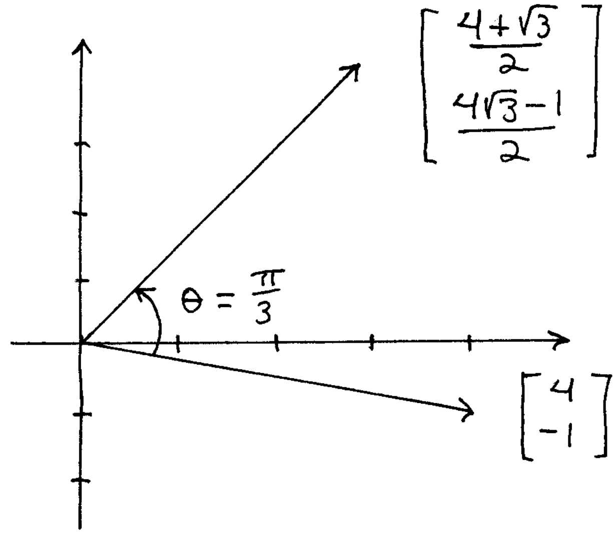
so A is a rotation matrix, rotating vectors counterclockwise $\pi/2$ radians

39.) $A = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$

$= \begin{bmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \frac{\pi}{6} & \cos \pi/6 \end{bmatrix}$, so A is a rotation matrix, rotating vectors counterclockwise $\frac{\pi}{6}$ radians

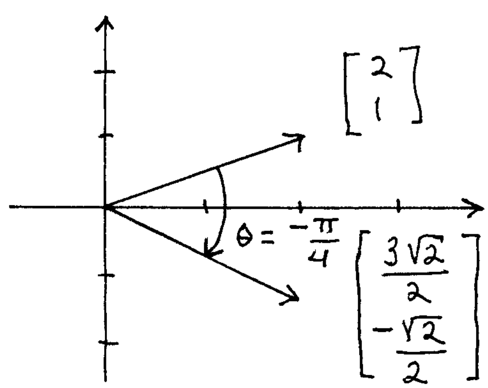
42.) $A = \begin{bmatrix} \cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\ \sin \frac{\pi}{3} & \cos \frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$,

then $\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{4+\sqrt{3}}{2} \\ \frac{4\sqrt{3}-1}{2} \end{bmatrix}$: (SEE graph.)



45.) $A = \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$,

then $\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$: (SEE graph.)



49.) $\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 0 & -1-\lambda \end{bmatrix}$

$= (2-\lambda)(-1-\lambda) - (3)(0) = -2 - 2\lambda + \lambda + \lambda^2$

$= \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0 \longrightarrow$

$\lambda = 2$ or $\lambda = -1$; solve $(A - \lambda I)X = 0$:

$\underline{\lambda_1 = 2}$: $\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -3 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow$ $x_2 = 0$ and $x_1 = t$ any # $\rightarrow$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so choose eigenvector

$\boxed{V_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$;

$\underline{\lambda_2 = -1}:$ $\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow$

$x_1 + x_2 = 0$, let $\underline{x_2 = t}$ any # $\rightarrow$

$x_1 = -t$, so

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so choose

eigenvector $\boxed{V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$

53.) $\det(A - \lambda I) = \det \begin{bmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{bmatrix}$

$= (-4-\lambda)(1-\lambda) - (-6)$

$= -4 + 4\lambda - \lambda + \lambda^2 + 6 = \lambda^2 + 3\lambda + 2$

$= (\lambda+1)(\lambda+2) = 0 \rightarrow \lambda = -1$ or $\lambda = -2$;

solve $(A - \lambda I)X = 0$:

$\underline{\lambda_1 = -1}:$ $\left[\begin{array}{cc|c} -3 & 2 & 0 \\ -3 & 2 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow$

$x_1 - \frac{2}{3}x_2 = 0 \rightarrow$ let $x_2 = t$ any # $\rightarrow$

$x_1 = \frac{2}{3}t \rightarrow X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2/3\, t \\ t \end{bmatrix} = \frac{1}{3}t \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

so choose eigenvector $\boxed{V_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}}$;

$\underline{\lambda_2 = -2}:$ $\left[\begin{array}{cc|c} -2 & 2 & 0 \\ -3 & 3 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow$

$x_1 - x_2 = 0 \to x_2 = t$ any # $\to x_1 = t \to$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so choose

eigenvector $\boxed{V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$.

55.) $\det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix}$

$= (2-\lambda)(3-\lambda) - 2 = 6 - 2\lambda - 3\lambda + \lambda^2 - 2$

$= \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4) = 0 \to$

$\lambda = 1$ or $\lambda = 4$; solve $(A - \lambda I)X = 0$:

$\underline{\lambda_1 = 1}$: $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \sim \begin{array}{c} \begin{matrix} x_1 & x_2 & \end{matrix} \\ \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \to x_1 + x_2 = 0 \to$

$x_2 = t$ any # $\to x_1 = -t$, so

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ so choose

eigenvector $\boxed{V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$;

$\underline{\lambda_2 = 4}$: $\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \sim \begin{array}{c} \begin{matrix} x_1 & x_2 & \end{matrix} \\ \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \to$

$x_1 - \frac{1}{2}x_2 = 0 \to x_2 = t$ any # $\to x_1 = \frac{1}{2}t \to$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2\, t \\ t \end{bmatrix} = \frac{1}{2}t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ so choose

eigenvector $\boxed{V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$.

63.) $A = \begin{bmatrix} 2 & 4 \\ -2 & -3 \end{bmatrix} \to \operatorname{tr} A = 2 + (-3) = -1 < 0$ and

$\det A = (2)(-3) - (4)(-2) = 2 > 0$, so real parts of eigenvalues λ_1 and λ_2 are negative

65.) $A = \begin{bmatrix} 4 & 4 \\ -4 & -3 \end{bmatrix} \rightarrow \operatorname{tr} A = 4 + (-3) > 0$, so real parts of eigenvalues λ_1 and λ_2 are not both negative

71.) $A = \begin{bmatrix} -1 & 0 \\ 3 & 1 \end{bmatrix} \rightarrow$ Find eigenvectors:

$\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 0 \\ 3 & 1-\lambda \end{bmatrix}$

$= (-1-\lambda)(1-\lambda) - (0)(3) = -1 + \lambda - \lambda + \lambda^2$

$= \lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0 \rightarrow \lambda = 1$ or $\lambda = -1$;

solve $(A - \lambda I)X = \odot$:

$\underline{\lambda_1 = 1}$: $\left[\begin{array}{cc|c} -2 & 0 & 0 \\ 3 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow x_1 = 0$ and

$x_2 = t$ any #; then $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

so choose eigenvector $\boxed{V_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$;

$\underline{\lambda_2 = -1}$: $\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 3 & 2 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 2/3 & 0 \end{array}\right] \end{array} \rightarrow x_1 + \frac{2}{3} x_2 = 0$

$\rightarrow x_2 = t$ any # $\rightarrow x_1 = \frac{-2}{3} t$, then

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2/3\, t \\ t \end{bmatrix} = \frac{1}{3} t \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, so choose

eigenvector $\boxed{V_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}}$;

Solve $x_1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow$

$$\begin{bmatrix} 0 \\ x_1 \end{bmatrix} + \begin{bmatrix} -2x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \rightarrow$$

$-2x_2 = 2 \rightarrow \boxed{x_2 = -1}$ and $x_1 + 3x_2 = 0 \rightarrow$

$x_1 + 3(-1) = 0 \rightarrow \boxed{x_1 = 3}$; then

$$\begin{bmatrix} 2 \\ 0 \end{bmatrix} = (3)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ 3 \end{bmatrix} \rightarrow$$

$$A^{15}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = A^{15}\left\{(3)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$$

$$= A^{15}\cdot(3)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + A^{15}(-1)\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= (3)A^{15}\begin{bmatrix} 0 \\ 1 \end{bmatrix} - A^{15}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= (3)\ (1)^{15}\begin{bmatrix} 0 \\ 1 \end{bmatrix} - (-1)^{15}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \end{bmatrix} - (-1)\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}.$$

72.) $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \rightarrow$ Find eigenvectors :

$$\det(A - \lambda I) = \det\begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix}$$

$$= (4-\lambda)(-1-\lambda) - (-3)(2)$$
$$= -4 - 4\lambda + \lambda + \lambda^2 + 6 = \lambda^2 - 3\lambda + 2$$
$$= (\lambda - 2)(\lambda - 1) = 0 \rightarrow \lambda = 1 \text{ or } \lambda = 2 \ ;$$

solve $(A - \lambda I)X = 0$:

$\underline{\lambda_1 = 1}$: $\left[\begin{array}{cc|c} 3 & -3 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow$

$x_1 - x_2 = 0 \rightarrow x_2 = t$ any # $\rightarrow x_1 = t$, then

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so choose

eigenvector $\boxed{V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$;

$$\underline{\lambda_2 = 2} : \begin{bmatrix} 2 & -3 & | & 0 \\ 2 & -3 & | & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 1 & -3/2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \end{array} \rightarrow$$

$x_1 - \frac{3}{2}x_2 = 0 \rightarrow x_2 = t$ any # $\rightarrow$

$x_1 = \frac{3}{2}t$, then $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}t \\ t \end{bmatrix} = \frac{1}{2}t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$,

so choose eigenvector $\boxed{V_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$;

Solve $x_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \rightarrow$

$$\begin{bmatrix} x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix} \rightarrow \begin{bmatrix} x_1 + 3x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

$$\rightarrow \left.\begin{array}{l} x_1 + 3x_2 = -4 \\ x_1 + 2x_2 = -2 \end{array}\right\} \rightarrow x_1 = -3x_2 - 4 \rightarrow$$

$(-3x_2 - 4) + 2x_2 = -2 \rightarrow$

$-x_2 = 2 \rightarrow \boxed{x_2 = -2}$ and $x_1 = -3(-2) - 4$

$\rightarrow \boxed{x_1 = 2}$; then

$$\begin{bmatrix} -4 \\ -2 \end{bmatrix} = (2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 3 \\ 2 \end{bmatrix} \rightarrow$$

$$A^{30}\begin{bmatrix} -4 \\ -2 \end{bmatrix} = A^{30}\left\{(2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$$

$$= A^{30}(2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + A^{30}(-2)\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= (2)A^{30}\begin{bmatrix}1\\1\end{bmatrix} + (-2)A^{30}\begin{bmatrix}3\\2\end{bmatrix}$$
$$= (2)(1)^{3}\begin{bmatrix}1\\1\end{bmatrix} + (-2)(2^{30})\begin{bmatrix}3\\2\end{bmatrix}$$
$$= \begin{bmatrix}2\\2\end{bmatrix} - 2^{31}\begin{bmatrix}3\\2\end{bmatrix}$$
$$= \begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}3\cdot 2^{31}\\2^{32}\end{bmatrix} = \begin{bmatrix}2-3\cdot 2^{31}\\2-2^{32}\end{bmatrix}$$

75.) $L = \begin{bmatrix}2 & 4\\0.3 & 0\end{bmatrix}$ then

$$\det(L-\lambda I) = \det\begin{bmatrix}2-\lambda & 4\\0.3 & 0-\lambda\end{bmatrix}$$
$$= (2-\lambda)(-\lambda) - (4)(0.3) = \lambda^2 - 2\lambda - 1.2 = 0 \rightarrow$$
$$\lambda = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1.2)}}{2(1)} = \frac{2 \pm \sqrt{8.8}}{2}$$
$$= \frac{2 \pm 2\sqrt{2.2}}{2} = 1 \pm \sqrt{2.2} \rightarrow$$

a.) $\lambda_1 = 1 + \sqrt{2.2}$ and $\lambda_2 = 1 - \sqrt{2.2}$

b.) Since $\lambda_1 = 1 + \sqrt{2.2} > 1$, then the total population size increases over time.

c.) Find eigenvector for $\lambda_1 = 1 + \sqrt{2.2} \approx 2.483$:

Solve $(L - \lambda I)X = 0$:

$$\left[\begin{array}{cc|c}-0.483 & 4 & 0\\0.3 & -2.483 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & -8.282 & 0\\0.3 & -2.483 & 0\end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c}1 & -8.282 & 0\\0 & 0 & 0\end{array}\right] \end{array} \rightarrow x_1 - 8.282\,x_2 = 0$$

$\rightarrow x_2 = t$ any # $\rightarrow x_1 = 8.282t$, then

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8.282t \\ t \end{bmatrix} = t \begin{bmatrix} 8.282 \\ 1 \end{bmatrix} \rightarrow$$

$\boxed{\begin{bmatrix} 8.282 \\ 1 \end{bmatrix}}$ is stable age distribution:

then $\frac{8.282}{8.282+1} \approx 89.2\%$ are 0-year olds and 10.8% are 1-year olds.

79.) $L = \begin{bmatrix} 0 & 5 \\ 0.09 & 0 \end{bmatrix}$ then

$\det(L - \lambda I) = \det \begin{bmatrix} 0-\lambda & 5 \\ 0.09 & 0-\lambda \end{bmatrix}$

$= (-\lambda)(-\lambda) - (5)(0.09) = \lambda^2 - 0.45 = 0 \rightarrow$

a.) $\lambda = \pm\sqrt{0.45} \rightarrow \lambda_1 = \sqrt{0.45}$ and $\lambda_2 = -\sqrt{0.45}$

b.) Since $\lambda = \sqrt{0.45} \approx 0.671 < 1$, then the total population size decreases over time.

c.) Solve $(L - \lambda I)X = 0$:

$$\begin{bmatrix} -0.671 & 5 & | & 0 \\ 0.09 & -0.671 & | & 0 \end{bmatrix} \sim \begin{array}{c} \begin{array}{cc} x_1 & \quad x_2 \end{array} \\ \begin{bmatrix} 1 & -7.452 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \end{array} \rightarrow$$

$x_1 - 7.452x_2 = 0 \rightarrow x_2 = t$ any # $\rightarrow$ $x_1 = 7.452t$, then

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7.452t \\ t \end{bmatrix} = t \begin{bmatrix} 7.452 \\ 1 \end{bmatrix}$$

$\longrightarrow \begin{bmatrix} 7.452 \\ 1 \end{bmatrix}$ <u>is stable age distribution</u> :

then $\dfrac{7.452}{7.452+1} \approx 88.2\%$ are 0-year olds

and 11.8 % are 1-year olds.