Section 2.5

16.) \[ Y = \frac{x+3}{x^2 - 3x - 10} \] \[ Y = x+3 \text{ and } Y = x^2 - 3x - 10 \text{ are continuous for all values of } x \text{ since they are polynomials.} \]

Therefore, since \[ Y = \frac{x+3}{x^2 - 3x - 10} \] is the quotient of these functions, it is continuous for all values of \( x \) except where \( x^2 - 3x - 10 = (x-5)(x+2) = 0 \), i.e., except for \( x = 5 \) and \( x = -2 \).

20.) \[ Y = \frac{x+2}{\cos x} \] \[ Y = x+2 \text{ is continuous for all values of } x \text{ since it is a polynomial.} \]

\[ Y = \cos x \text{ is continuous for all values of } x \text{ since it is a well-known trig function.} \]

Therefore, since \[ Y = \frac{x+2}{\cos x} \] is the quotient of these functions, it is continuous for all values of \( x \) except where \( \cos x = 0 \), i.e., except for \( x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2} \),...

26.) \[ Y = (3x-1)^\frac{1}{4} \] \[ \text{let } f(x) = x^{\frac{1}{4}} \text{ which is continuous for } x \geq 0, \text{ and let } g(x) = 3x-1, \text{ which is continuous for all values of } x \text{ since it is a polynomial;} \]

since \[ Y = (3x-1)^{\frac{1}{4}} = f(3x-1) = f(g(x)) \]
is functional composition, it is
continuous for all $x$-values for which $3x - 1 \geq 0$, i.e., for $x \geq \frac{1}{3}$.

42.) \[ g(x) = \frac{x^2 - 16}{x^2 - 3x - 4} \]

\[ \lim_{x \to 4} g(x) = \lim_{x \to 4} \frac{x^2 - 16}{x^2 - 3x - 4} \]

\[ = \lim_{x \to 4} \frac{(x-4)(x+4)}{(x-4)(x+1)} = \frac{8}{5}, \]

so define $g(4) = \frac{8}{5}$ and $g$ will be continuous at $x = 4$.

43.) Let \[ f(x) = \begin{cases} 
  x^2 - 1, & \text{if } x < 3 \\
  2ax, & \text{if } x \geq 3
\end{cases} \]

\[ Y = x^2 - 1 \text{ is continuous for } x < 3 \text{ (polynomial)}; \]

\[ Y = 2ax \text{ is continuous for } x > 3 \text{ (line)}. \]

Make $f$ continuous at $x = 3$ by forcing limits to be equal:

\[ \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2 - 1) = 9 - 1 = 8, \]

\[ \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2ax) = 6a, \text{ so} \]

\[ 6a = 8 \rightarrow a = \frac{4}{3}. \]
44.) Let \( g(x) = \begin{cases} x, & \text{if } x < -2 \\ bx^2, & \text{if } x \geq -2 \end{cases} \)

\( Y = x \) is continuous for \( x < -2 \) (line),
\( Y = bx^2 \) is continuous for \( x > -2 \) (parabola); make \( g \) continuous at \( x = -2 \) by forcing limits to be equal:

\[ \lim_{x \to -2^-} g(x) = \lim_{x \to -2^-} x = -2, \]
\[ \lim_{x \to -2^+} g(x) = \lim_{x \to -2^+} bx^2 = 4b, \]
so \( 4b = -2 \implies b = -\frac{1}{2} \).

I.) Prove \( x^3 = x + 2 \) is solvable:
\( x^3 = x + 2 \implies x^3 - x - 2 = 0 \), so let
\( f(x) = x^3 - x - 2 \) and \( m = 0 \); note that \( f(1) = -2 < 0 \) and \( f(2) = 4 > 0 \)
so \( m = 0 \) is between \( f(1) \) and \( f(2) \);
use the interval \([1, 2]\); \( f \) is a continuous function on \([1, 2]\) since
it is a polynomial. By the IVT it follows that there is a number \( c, \ 1 \leq c \leq 2 \), so that \( f(c) = m \), i.e.,
\( c^3 - c - 2 = 0 \), and the original equation is solvable.
II.) Prove $2^x + \sin x = x$ is solvable:

$2^x + \sin x = x \Rightarrow 2^x - x + \sin x = 0$

Let $f(x) = 2^x - x + \sin x$ and $m = 0$; $f$ is continuous for all values of $x$ since it is the sum of continuous functions ($y = 2^x$, a line, and $y = \sin x$, a well-known trig function); note that $f(0) = 2 > 0$ and $f(\pi) = 2 - \pi - \sin \pi = 2 - \pi < 0$, so $m = 0$ is between $f(0)$ and $f(\pi)$.

Use the interval $[0, \pi]$. By the IMVT it follows that there is a number $c$, $0 \leq c \leq \pi$, so that $f(c) = m$, i.e.,

$2^c - c + \sin c = 0$, and the original equation is solvable.
Worksheet 1

1.) a.) \( f(x) = \begin{cases} x^2 + 3 & \text{if } x \neq -1 \\ 2 & \text{if } x = -1 \end{cases} \) \at \( x = -1 \):

i.) \( f(-1) = 2 \)

ii.) \( \lim_{x \to -1} f(x) = \lim_{x \to -1} (x^2 + 3) = 1 + 3 = 4 \)

iii.) \( \lim_{x \to -1} f(x) = 4 \neq 2 = f(-1) \) \so

\( f \) is NOT continuous \at \( x = -1 \)

b.) \( g(x) = \begin{cases} x + 1 & \text{if } x \geq 0 \\ 2 - x^2 & \text{if } x < 0 \end{cases} \) \at \( x = 0 \):

i.) \( g(0) = 0 + 1 = 1 \)

ii.) \( \lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x + 1) = 1 \);

\( \lim_{x \to 0^-} g(x) = \lim_{x \to 0} (2 - x^2) = 2 \) \so

\( \lim_{x \to 0} g(x) \) DNE \; \therefore

\( g \) is NOT continuous \at \( x = 0 \)

c.) \( f(x) = \begin{cases} x - 2 & \text{if } x > 1 \\ 0 & \text{if } x = 1 \\ -x & \text{if } x < 1 \end{cases} \) \at \( x = 1 \):

i.) \( f(1) = 0 \)

ii.) \( \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x - 2) = -1 \) \so
\[
\lim_{x \to 1^{-}} f(x) = \lim_{x \to 1^{-}} (-x) = -1 \quad \text{so}
\]
\[
\lim_{x \to 1} f(x) = -1
\]

iii.) \(\lim_{x \to 1} f(x) = -1 \neq 0 = f(1)\), so

\(f\) is NOT continuous at \(x = 1\).

2.) a.) \(f(x) = x^5 + x^4 + x^3 + x^2 + x + 1\) is continuous for all values of \(x\) since it is a polynomial.

b.) \(g(x) = \frac{\sin x}{x^2 + 4}\); \(Y = \sin x\) (well known) and \(Y = x^2 + 4\) (polynomial) are continuous for all values of \(x\); so the QUOTIENT \(g(x) = \frac{\sin x}{x^2 + 4}\) is continuous for all values of \(x\) since \(x^2 + 4 \neq 0\).

c.) \(f(x) = \frac{x + 3}{x^2 - 4}\); \(Y = x + 3\) and \(Y = x^2 - 4\) (polynomials) are continuous for all values of \(x\); the QUOTIENT \(f(x) = \frac{x + 3}{x^2 - 4}\) is continuous for all values of \(x\) except where \(x^2 - 4 = 0\), i.e., except when \(x = \pm 2\).
d.) \( g(x) = \cos(x^3 - x) \); \( f(x) = \cos x \) (well known) and \( h(x) = x^3 - x \) (polynomial) are continuous for all values of \( x \); so the composition \( f(h(x)) = f(x^3 - x) = \cos(x^3 - x) \) is continuous for all values of \( x \).

3.) a.) \( f(x) = \begin{cases} \frac{x^2 - 7x + 6}{x - 6} & \text{if } x \neq 6 \\ A & \text{if } x = 6 \end{cases} \)

\[
\lim_{x \to 6} f(x) = \lim_{x \to 6} \frac{x^2 - 7x + 6}{x - 6} = \lim_{x \to 6} \frac{(x-6)(x-1)}{x-6} = 6-1 = 5 \quad \text{and} \quad f(6) = A \quad \text{so choose } A = 5
\]

b.) \( f(x) = \begin{cases} A^2 x - A & \text{if } x \geq 1 \\ 2 & \text{if } x < 1 \end{cases} \)

Require that \( \lim_{x \to 1^+} f(x) = 2 \),

\[
\lim_{x \to 1^+} (A^2 x - A) = 2 \quad \Rightarrow \quad A^2 - A = 2 \quad \Rightarrow \quad A^2 - A - 2 = 0 \quad \Rightarrow \quad (A-2)(A+1) = 0 \quad \Rightarrow \quad A = 2 \text{ or } A = -1
\]

c.) \( f(x) = \begin{cases} \frac{A+x}{A+1} & \text{if } x < 0 \\ A x^3 + 3 & \text{if } x \geq 0 \end{cases} \)
cubic

\[ \frac{A + 0}{A + 1} = 3 \rightarrow A = 3A + 3 \rightarrow -3 = 2A \rightarrow A = -\frac{3}{2} \]

d.) \[ f(x) = \begin{cases} 
3 & \text{if } x \leq 1 \\
A x^2 + B & \text{if } 1 < x \leq 2 \\
5 & \text{if } x > 2 
\end{cases} \]

Require Two conditions:

i.) \[ \lim_{x \to 1^+} f(x) = 3 \rightarrow \lim_{x \to 1^+} (A x^2 + B) = 3 \rightarrow A + B = 3 \] AND

ii.) \[ \lim_{x \to 2^-} f(x) = 5 \rightarrow \lim_{x \to 2^-} (A x^2 + B) = 5 \rightarrow 4A + B = 5 \]

\[ B = 3 - A \rightarrow (\text{sub } B) \rightarrow 4A + (3 - A) = 5 \rightarrow 3A = 2 \rightarrow A = \frac{2}{3}, \quad B = \frac{7}{3} \]

e.) \[ f(x) = \begin{cases} 
A x - B & \text{if } x \leq -1 \\
2x + 3A + B & \text{if } -1 < x \leq 1 \\
4 & \text{if } x > 1 
\end{cases} \]
Require **Two** conditions:

i.) \( \lim_{x \to 1^-} f(x) = 4 \rightarrow \)
\[ \lim_{x \to 1^-} (2x + 3A + B) = 4 \rightarrow \]
\[ 2 + 3A + B = 4 \rightarrow 3A + B = 2 \] AND

ii.) \( \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \rightarrow \)
\[ \lim_{x \to 1^-} (Ax - B) = \lim_{x \to 1^+} (2x + 3A + B) \rightarrow \]
\[ -A - B = -2 + 3A + B \rightarrow 2 = 4A + 2B \rightarrow 2A + B = 1 \] ; solve system.

3A + B = 2 \rightarrow B = 2 - 3A \rightarrow (3) \rightarrow

2A + (2 - 3A) = 1 \rightarrow -A = -1 \rightarrow

\( A = 1, \quad B = -1 \)