The Derivative

\[ y = f(x) \]

Secant line with slope \( \frac{f(x) - f(a)}{x - a} \)

\[ \Delta f = f(x) - f(a) \]

Tangent line with slope \( f'(a) \)

\[ \Delta x = \text{run} \]

Note that the slope of the secant line is

\[ \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x) - f(a)}{x - a} \]

We can obtain slope of tangent line by letting \( \Delta x \) get 'small' (i.e. \( x \to a \)) which leads to the following:

**Defn.** Let \( f \) be a function on \((b,c)\). The derivative of \( f \) at the point \( a \in (b,c) \) is

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

provided this limit exists and is finite.

**Notes:**

1) If \( f'(a) \) exists and is finite, we say \( f \) is differentiable at \( a \). Otherwise, \( f \) is not differentiable at \( a \).
2) \( f'(a) \) is the slope of the tangent line through point \((a,f(a))\).
3) \( \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \approx \frac{\Delta f}{\Delta x} \) can be thought of as `infinitesimal division' (i.e. division of two `small' numbers).
4) Other `disguises' include: \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h} \)

**Thm 28.2**

If \( f \) is differentiable at \( a \), then \( f \) is continuous at \( a \).

Moral: The set of differentiable functions is a subset of continuous functions.
Thm (28.3)

Let \( f \) and \( g \) be differentiable functions at \( x = a \). Then the functions \( cf \) (with \( c \) a constant), \( f + g \), \( fg \), and \( \frac{f}{g} \) are differentiable at \( x = a \) when defined.

The derivatives are:

1) \((cf)'(a) = c \cdot f'(a)\) \quad \text{(Constant Multiple Rule)}
2) \((f + g)'(a) = f'(a) + g'(a)\) \quad \text{(Sum Rule)}
3) \((fg)'(a) = f(a)g'(a) + f'(a)g(a)\) \quad \text{(Product Rule)}
4) \(\left(\frac{f}{g}\right)'(a) = \frac{f(a)g'(a) - f'(a)g(a)}{g(a)^2}\) \quad \text{(Quotient Rule)}

Note: Clearly the quotient rule only makes sense if \( g(a) \neq 0 \).

Chain Rule (28.4)

If \( f \) is differentiable at \( a \) and \( g \) is differentiable at \( f(a) \), then \((g \circ f)'(a)\) is differentiable at \( a \). Moreover, 
\[ (g \circ f)'(a) = g'(f(a)) \cdot f'(a). \]

Derivative Rules for Well-Known Functions

 e) \((x^n)' = nx^{n-1}\) for \( n \in \mathbb{R} \) \quad \text{(Power Rule)}
 f) \((c)' = 0\) for \( c \in \mathbb{R} \)
 g) \((\sin x)' = \cos x\)
 h) \((\cos x)' = -\sin x\)
 i) \((b^x)' = b^x \ln b\) for \( b \in \mathbb{R} \)
 j) \((\log_b x)' = \frac{1}{x \ln b}\) for \( b \in \mathbb{R} \)
 k) \((\tan x)' = \sec^2 x\)
 l) \((\sec x)' = \sec x \tan x\)

Note: You may use these facts on a test unless specifically asked to prove it using the definition of the derivative.