Defn. Let \( a \in \mathbb{R} \) and \( L \in \mathbb{R} \cup \{ \pm \infty \} \).

a) The \( \text{limit of } f \text{ as } x \text{ approaches } a \text{ equals } L \), written \( \lim_{x \to a} f(x) = L \), if for all sequences \( \{x_n\} \subseteq (c,a) \cup (a,b) \) with \( x_n \to a \), \( n \to \infty \), \( f(x_n) = L \).

b) The \( \text{left-handed limit } \lim_{x \to a^-} f(x) = L \) if for all sequences \( \{x_n\} \subseteq (c,a) \) with \( x_n \to a \), \( n \to \infty \), \( f(x_n) = L \).

c) The \( \text{right-handed limit } \lim_{x \to a^+} f(x) = L \) if for all sequences \( \{x_n\} \subseteq (a,b) \) with \( x_n \to a \), \( n \to \infty \), \( f(x_n) = L \).

d) \( \lim_{x \to \infty} f(x) = L \) if for all sequences \( \{x_n\} \subseteq (c,\infty) \) with \( x_n \to \infty \), \( n \to \infty \), \( f(x_n) = L \).

e) \( \lim_{x \to -\infty} f(x) = L \) if for all sequences \( \{x_n\} \subseteq (-\infty,b) \) with \( x_n \to -\infty \), \( n \to \infty \), \( f(x_n) = L \).

Notes:
1) These definitions allow us to make formal those concepts we learned in math 21A using concepts (i.e., limits of sequences) we studied earlier in the book.
2) Obviously, \( b, c \in \mathbb{R} \) and all intervals \( \text{MUST be in domain of } f \).
3) If any of these limits exists, there are unique and independent of the choice of interval (i.e., does not depend on choice of \( b, c \)).
**Thm (20.4, 20.5)**

Let $f$ and $g$ be functions where $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$ exist and are finite. Then

1) $\lim_{x \to a} (f+g)(x) = L_1 + L_2$

2) $\lim_{x \to a} (f \cdot g)(x) = L_1 \cdot L_2$

3) $\lim_{x \to a} \left( \frac{f}{g} \right)(x) = \frac{L_1}{L_2}$ as long as $g(x) \neq 0$ for all $n$ and $L_2 \neq 0$.

4) If $g$ is defined on range of $f$ and $f$ is continuous, then $\lim_{x \to a} (g \circ f)(x) = g(L)$, provided $g$ is also defined at $L$.

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**Thm (20.6)**

$\lim_{x \to a} f(x) = L$ for $L$ finite if and only if

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x$ with $|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

- **Notes:**
  1) I'll refer to (1) as the '3δ-property' for limits.
  2) Many textbooks, including our math 21A book, use (1) as the definition of the limit, which is equivalent to ours because of this theorem.
  3) For left-handed limits, $\lim_{x \to a^-} f(x) = L$, this becomes
     $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x$ with $a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon$.
  4) For right-handed limits $\lim_{x \to a^+} f(x) = L$, this becomes
     $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\forall x$ with $a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon$.

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**Thm (20.10)**

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.

- **Note:** I will refer to the right hand side as the 'kiss condition'.

- **Moral:** A limit existing is equivalent to both the left and right limits existing and matching.