Defn f is uniformly continuous on $S \subseteq \mathbb{R}$ if

1) $\forall \delta > 0$, $\exists \varepsilon > 0$ such that $\forall x, y \in S$ with $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. It is called uniformly continuous if it's uniformly continuous on its domain.

-Notes: 1) Unlike continuity, uniform continuity cannot be defined at a point, only on a set.
2) Uniform continuity on $S$ implies continuity on $S$, which is easily shown by letting $y = x_0$ and comparing to $\varepsilon$-$\delta$-property.
3) The negation of (1) is $\exists \delta > 0, \forall \varepsilon > 0, \exists x, y \in S$ with $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

Thm (19.2)
If $f$ is continuous on $[a, b]$, then $f$ is uniformly continuous on $[a, b]$.

-Notes: 1) With the continuity theorems (17.3-17.5) and list of known continuous $\mathbb{R}$ns, we can show most functions are uniformly continuous on any $[a, b]$ with this theorem.
2) The interval $[a, b]$ can be replaced with any closed and bounded set $S$, and the theorem still holds.

Moral: On closed and bounded sets, continuity and uniform continuity are equivalent.

Thm (19.4)
If $f$ is uniformly continuous on $S$ and $\{s_n\}$ is a Cauchy sequence in $S$, then $\{f(s_n)\}$ is a Cauchy sequence.
- Note: This theorem gives us a way to prove/disprove uniform continuity using sequences and is the closest analog to the 'sequential form of continuity'.
- Moral: Uniform continuity preserves Cauchy sequences.

Defn The function \( \tilde{f} \) is an extension of a function \( f \) if \( \text{domain}(f) \subseteq \text{domain}(\tilde{f}) \) and \( f(x) = \tilde{f}(x) \quad \forall x \in \text{domain}(f) \).

Thm (19.5)

A function \( f \) is uniformly continuous on \( (a, b) \) if and only if \( f \) can be extended to a continuous function \( \tilde{f} \) on \( [a, b] \).

- Note: Very useful theorem for showing:
  a) Continuous functions with removable singularities (a.k.a. 'holes') are uniformly continuous once 'filled-in' correctly. (i.e. \( f(x) = \frac{\sin(x)}{x} \))
  b) Continuous functions with an undefined limits through oscillation are not uniformly continuous. (i.e. \( f(x) = \sin(\frac{1}{x}) \)).

Thm (19.6)

Let \( f \) be continuous on interval \( I \), and let \( I^0 \) be all points in \( I \) without the endpoints. If \( f \) is differentiable on \( I^0 \) along with \( f' \) bounded on \( I^0 \), then \( f \) is uniformly continuous on \( I \).

- Note: \( I^0 \) is sometimes referred to as the interior of \( I \) and is an open set (i.e. \( I = [a, b] \Rightarrow I^0 = (a, b) \)).
- Moral: A continuous function with bounded slope on its interior is uniformly continuous.