Defn  A sequence of functions $\{f_n\}$ converges pointwise to a function $f$, written $\{f_n\} \rightarrow f$, on $S \subseteq \mathbb{R}$ if
\[
\lim_{n \to \infty} f_n(x) = f(x) \quad \forall x \in S.
\]

Defn  A sequence of functions $\{f_n\}$ converges uniformly to a function $f$, written $\{f_n\} \Rightarrow f$, on $S \subseteq \mathbb{R}$ if
\[
\forall \varepsilon > 0, \exists N \text{ such that } \forall x \in S, \forall n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon
\]

Notes: 1) Uniform convergence is stronger than pointwise convergence (i.e. $\{f_n\} \Rightarrow f \Rightarrow \{f_n\} \rightarrow f$).
2) If $\{f_n\} \Rightarrow g_1$ and $\{f_n\} \rightarrow g_2 \Rightarrow g_1(x) = g_2(x)$ (i.e. the pointwise and uniform limits are unique). Hence, when trying to find the uniform limit one finds the pointwise limit as a candidate.

Thm (24.3)
Let $\{f_n\}$ be a sequence of functions defined on $S \subseteq \mathbb{R}$ where $\{f_n\} \Rightarrow f$ on $S$. If $f_n$ is continuous at $x_0$ for all $n$, then $f$ is continuous at $x_0$.

Note: This gives us a quick way of determining whether the limit of continuous functions is uniform by checking whether the limit function is continuous, assuming we can find the limit function.

Moral: The uniform limit of continuous functions is continuous.

Prop (24.4)
$\{f_n\} \Rightarrow f$ on $S \subseteq \mathbb{R}$ if and only if $\lim_{n \to \infty} [\sup \{f(x) - f_n(x) : x \in S\}] = 0$.

Note: Another useful way of determining uniform convergence.

Moral: A sequence of functions converging uniformly is equivalent to the biggest difference between a sequence element and the limit function tends to zero.
Defn
A sequence of functions \( \{f_n\} \) defined on \( S \subseteq \mathbb{R} \) is uniformly Cauchy if
\[
\forall \varepsilon > 0, \exists N \text{ such that } \forall x \in S, \forall m,n > N \implies |f_n(x) - f_m(x)| < \varepsilon
\]

-Notes: 1) A useful concept because it allows us to show uniform convergence without knowing the limit function.
2) It's easy to show uniform convergence \( \implies \) uniform Cauchy.

Thm (25.4)
Let \( \{f_n\} \) be a sequence of functions that are uniformly Cauchy on \( S \subseteq \mathbb{R} \). Then there exists a function \( f \) such that \( \{f_n\} \to f \) on \( S \).

Defn
Let \( \{g_k\} \) be a sequence of functions. The resulting series \( \sum_{k=0}^{\infty} g_k(x) \) is called a series of functions, which makes sense if the sequence of partial sums \( f_n(x) := \sum_{k=0}^{n} g_k(x) \) converges or diverges to \( \pm \infty \) pointwise. If there exists an \( f \) such that \( \{f_n\} \to f \) on \( S \subseteq \mathbb{R} \), then we say the series is uniformly convergent on \( S \).

-Note: If \( g_k \) is continuous for all \( k \) and \( \sum g_k \) is uniformly convergent on \( S \), then \( f(x) := \sum g_k(x) \) is continuous on \( S \) by Thm 24.3.

Defn
The series of functions \( \sum g_k \) satisfies the Cauchy criterion if
\[
\forall \varepsilon > 0, \exists N \text{ such that } \forall x \in S, \forall n,m > N \implies |\sum_{k=m}^{n} g_k(x)| < \varepsilon.
\]

-Notes: 1) This is equivalent to the seq. of partial sums being uniformly Cauchy.
2) The series satisfies the Cauchy criterion if it's uniformly convergent.

Weierstrass M-Test
Consider sequence \( \{M_k\} \subseteq \mathbb{R}_{>0} \) with \( \sum M_k < \infty \).
If \( |g_k(x)| \leq M_k \ \forall x \in S \subseteq \mathbb{R} \), then \( \sum g_k \) converges uniformly on \( S \).