18.2) It would breakdown because the limit of subsequences \( x_0 \) and \( y_0 \) does not have to be in the interval \((a, b)\). Since we would only assume continuity of \( f \) on \((a, b)\), we would lose it at the endpoints \( a \) and \( b \). This could cause the function to be unbounded and not contain it’s minimum and/or maximum. For example, take \( f(x) = \frac{1}{x} \) on \((0, 1)\).

18.4) Suppose \( S \subseteq \mathbb{R} \) and there exists a sequence \( \{x_n\} \) in \( S \) that converges to \( x_0 \notin S \). Consider the function \( f(x) = \frac{1}{x - x_0} \) on \( S \). \( f \) is continuous on \( S \) since \( x_0 \notin S \) by the fact \( x - x_0 \) is a polynomial and the division law of continuity (Theorem 17.4). \( f \) is unbounded because of the division by zero that occurs within the sequence \( f(x_n) \) (Theorem 9.10 and Exercise 9.10b in the case the sequence is negative).

18.6) We can rewrite the equation \( x = \cos x \) as \( x - \cos x = 0 \). Let \( f(x) = x - \cos x \) and \( m = 0 \). Notice that \( f(0) = -1 < 0 \) and \( f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} > 0 \). Choose interval \((0, \frac{\pi}{2})\), so that \( f(0) \leq m = 0 < f\left(\frac{\pi}{2}\right) \). By the Intermediate Value Theorem, there exists at least one \( x \in (0, \frac{\pi}{2}) \) with \( f(x) = m = 0 \). With this \( x \), we have

\[
 f(x) = 0 \iff x - \cos x = 0 \iff x = \cos x.
\]

Hence, \( x = \cos x \) for \( x \in (0, \frac{\pi}{2}) \).

18.8) Suppose \( f \) is a continuous function on \( \mathbb{R} \) and \( f(a)f(b) < 0 \) for some \( a, b \in \mathbb{R} \). Since \( f(a)f(b) < 0 \), one of the two values \( f(a) \) or \( f(b) \) must be positive while the other is negative. Without loss of generality, assume \( f(a) < 0 \) and \( f(b) > 0 \). Let \( m = 0 \) and choose interval \((a, b)\) since \( f(a) < m = 0 < f(b) \). By the Intermediate Value Theorem, there exists at least one \( x \in (a, b) \) with \( f(x) = m = 0 \), proving the claim.

18.10) Suppose \( f \) is continuous on \([0, 2]\) and \( f(0) = f(2) \). Define \( g(x) = f(x + 1) - f(x) \) on \([0, 1]\). Notice \( g(0) = f(2) - f(1) \) and \( g(1) = f(1) - f(0) = f(1) - f(2) \) by assumption. So we have \( g(0) = -g(1) \). We consider two cases: a) \( g(0) = 0 \) or b) \( g(0) \neq 0 \).

Case a) If \( g(0) = 0 \), this implies \( f(2) - f(1) = 0 \) and consequently \( f(2) = f(1) \). So for \( x = 2 \) and \( y = 1 \), we have \(|x - y| = 1\) and \( f(x) = f(y) \), proving the claim.

Case b) If \( g(0) \neq 0 \), without loss of generality we can assume \( g(0) < 0 \), which implies \( g(1) > 0 \). \( g \) is continuous since \( f \) is continuous and composition and differences preserve continuity. Let \( m = 0 \), and choose interval \((0, 1)\) since \( g(0) < m = 0 < g(1) \). By the Intermediate Value Theorem, there exists at least one \( c \in (0, 1) \) with \( g(c) = m = 0 \). With this \( c \), we have

\[
 g(c) = 0 \iff f(c + 1) - f(c) \iff f(c + 1) = f(c).
\]

So for \( x = c + 1 \) and \( y = c \), we have \(|x - y| = 1\) and \( f(x) = f(y) \), proving the claim.