

1.

- (a) State the PRECISE definition of the limit for

$$\lim_{x \rightarrow a} f(x) = L$$

- Given any number $\varepsilon > 0$, there exists another number $\delta > 0$ such that if $|x - a| < \delta$, then $|f(x) - L| < \varepsilon$

or
- $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

- (b) Give an
- ε, δ
- proof for the following limit. This is a writing exercise.

$$\lim_{x \rightarrow -2} (x^2 + 3) = 7$$

$\varepsilon > 0$ is given. Need to find $\delta > 0$ s.t.

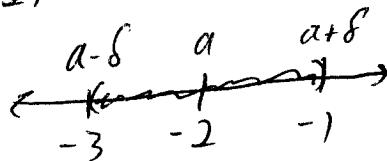
$$|x - (-2)| = |x + 2| < \delta \Rightarrow |f(x) - 7| < \varepsilon$$

Begin w/ $|f(x) - 7| < \varepsilon$ & solve for $|x + 2| < \delta$

$$|f(x) - 7| < \varepsilon \Leftrightarrow |x^2 + 3 - 7| < \varepsilon \Leftrightarrow |x^2 - 4| < \varepsilon$$

$$\Leftrightarrow |x - 2||x + 2| < \varepsilon. \text{ Want to remove } |x - 2|$$

Assume $\delta \leq 1$,



$$\Rightarrow -3 < x < -1 \Rightarrow -5 < x - 2 < -3 \Rightarrow 3 < |x - 2| < 5.$$

$$\text{Then } |x - 2||x + 2| < 5|x + 2| < \varepsilon \Leftrightarrow |x + 2| < \frac{\varepsilon}{5}$$

$$\text{Choose } \delta = \min \left\{ \frac{\varepsilon}{5}, 1 \right\}.$$

Hence, if $|x + 2| < \delta$, it follows that $|f(x) - 7| < \varepsilon$

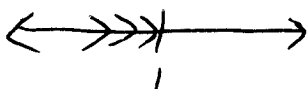
2. Determine the following limits

$$(a) \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - 1} = \frac{0}{0} = \lim_{x \rightarrow 1} \frac{(x+2)(\cancel{x-1})}{(x+1)(\cancel{x-1})} = \frac{3}{2}$$

$$(b) \lim_{h \rightarrow 0} \frac{\sin 5h}{2h} = \frac{0}{0} = \frac{5}{2} \cdot \lim_{h \rightarrow 0} \frac{\sin \cancel{5} h}{\cancel{5} h} = \frac{5}{2}$$

$\downarrow$
1

$$(c) \lim_{x \rightarrow 1^-} \frac{\sqrt{2x}(x-1)}{|x-1|} = \frac{0}{0} = \lim_{x \rightarrow 1^-} \frac{\sqrt{2x}(\cancel{x-1})}{-(\cancel{x-1})} = -\sqrt{2}$$



$$(d) \lim_{x \rightarrow 4} \frac{\sqrt{5+x}-3}{x-4} = \frac{0}{0} = \lim_{x \rightarrow 4} \frac{\sqrt{5+x}-3}{x-4} \cdot \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$$

$$= \lim_{x \rightarrow 4} \frac{x+5-9}{(x-4)(\sqrt{x+5}+3)} = \lim_{x \rightarrow 4} \frac{\cancel{x-4}}{(\cancel{x-4})(\sqrt{x+5}+3)} = \frac{1}{6}$$

$$(e) \lim_{x \rightarrow \infty} \frac{3 + \cos(x)}{x^4} \quad (\text{HINT : Use the Squeeze Principle.})$$

$$-1 \leq \cos x \leq 1 \Rightarrow 2 \leq 3 + \cos x \leq 4$$

$$\Rightarrow \frac{2}{x^4} \leq \frac{3 + \cos x}{x^4} \leq \frac{4}{x^4} \quad \& \quad \lim_{x \rightarrow \infty} \frac{2}{x^4} = 0 = \lim_{x \rightarrow \infty} \frac{4}{x^4}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{3 + \cos x}{x^4} = 0 \quad \text{by squeeze principle.}$$

3. Sketch the following graph. Make sure to CLEARLY state intercepts, asymptotes, along with the domain and range.

$$y = \frac{x+2}{x^2(x+1)}$$

x-int: $y=0$ when $x=-2$

y-int: $x=0$ DNE

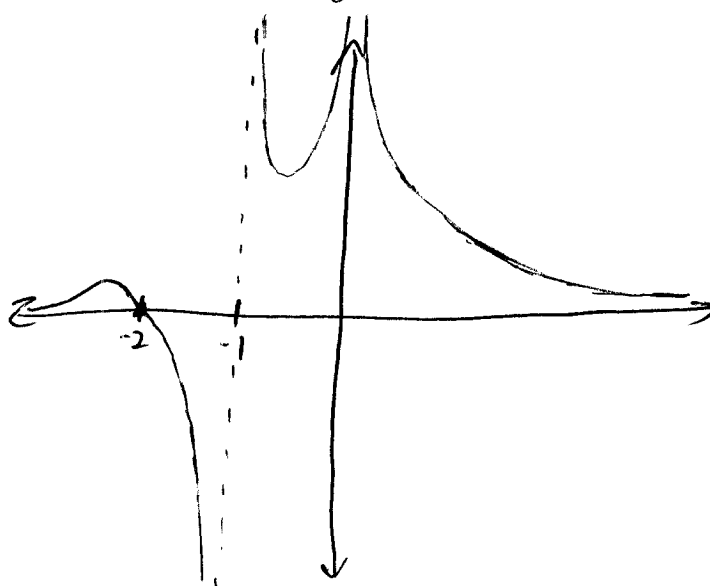
$$\text{V.A. } \left. \begin{array}{l} \lim_{x \rightarrow 0^-} y = +\infty \\ \lim_{x \rightarrow 0^+} y = +\infty \end{array} \right\} x=0$$

$$\left. \begin{array}{l} \lim_{x \rightarrow -1^-} y = -\infty \\ \lim_{x \rightarrow -1^+} y = +\infty \end{array} \right\} x=-1$$

$$\text{H.A. } \left. \begin{array}{l} \lim_{x \rightarrow \infty} \frac{x+2}{x^3+x^2} \cdot \frac{1/x^3}{1/x^3} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^2} + \frac{2}{x^3}}{1 + \frac{1}{x}} = 0 \\ \lim_{x \rightarrow -\infty} \frac{x+2}{x^3+x^2} \cdot \frac{1/x^3}{1/x^3} = \lim_{x \rightarrow -\infty} \frac{\frac{1}{x^2} + \frac{2}{x^3}}{1 + \frac{1}{x}} = 0 \end{array} \right\} y=0$$

Domain: $\forall x \quad x \neq 0, -1$

Range: $\forall y$



4. Use the Intermediate Value Theorem to show that the equation $x^3 = 10 + \sqrt{x}$ is solvable. This is a writing exercise.

$$x^3 = 10 + \sqrt{x} \Leftrightarrow x^3 - \sqrt{x} - 10 = 0$$

Let $f(x) = x^3 - \sqrt{x} - 10$ & $m = 0$,

where f is continuous (poly + sqrt is cont.)

Since $f(0) = -10 < 0$ & $f(9) = 9^3 - 3 - 10 > 0$

$\Rightarrow f(0) < m = 0 < f(9)$. Choose interval $[0, 9]$

By IMVT, there exists @ least one c , $0 \leq c \leq 9$,

s.t. $f(c) = 0 \Leftrightarrow c^3 = 10 + \sqrt{c}$. Thus, equation is solvable.

4.) Consider the following function $f(x) = \begin{cases} \frac{x^2 - 3x}{x^2 - 9}, & \text{if } x \neq 3, -3 \\ \frac{1}{2}, & \text{if } x = 3 \\ 0, & \text{if } x = -3 \end{cases}$

Determine if f is continuous at $x = 3$.

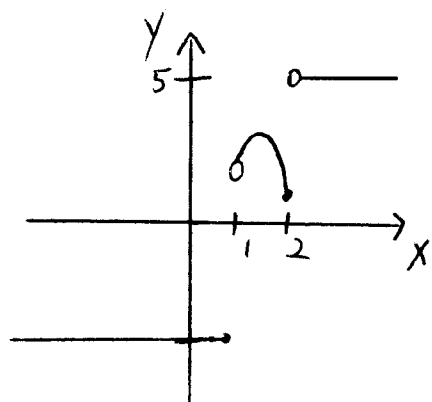
i.) $f(3) = \frac{1}{2}$
 ii.) $\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \frac{x^2 - 3x}{x^2 - 9} \stackrel{\frac{0}{0}}{=} \lim_{x \rightarrow 3} \frac{x(x-3)}{(x-3)(x+3)} = \frac{3}{6} = \frac{1}{2}$
 iii.) $\lim_{x \rightarrow 3} f(x) = f(3)$ so

f is continuous at $x = 3$.

4. Consider the following function

$$f(x) = \begin{cases} -4 & \text{if } x \leq 1 \\ Ax^2 + B & \text{if } 1 < x \leq 2 \\ 5 & \text{if } x > 2 \end{cases}$$

Use limits and a "fake graph" to determine the value of constants A and B so that the following function is continuous for all values of x .



We need for continuity ('kiss condition')

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) \quad \& \quad \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$$

$$\Rightarrow \lim_{x \rightarrow 1^+} Ax^2 + B = -4 \quad \& \quad \lim_{x \rightarrow 2^-} Ax^2 + B = 5$$

$$\Rightarrow \underline{A + B = -4} \quad \& \quad \underline{4A + B = 5}$$

$$\Rightarrow B = -4 - A \quad \xrightarrow{\text{sub}} \quad \underline{4A + B = 5}$$

$$\Rightarrow 4A - 4 - A = 5$$

$$\Rightarrow 3A = 9$$

$$\Rightarrow \boxed{A = 3 \quad \& \quad B = -4 - 3 = -7}$$

- 2.) a.) Determine the domain of $f(x) = \sqrt{7-x}$.

$$7-x \geq 0 \rightarrow 7 \geq x \text{ so}$$

Domain: all $x \leq 7$

- b.) Determine the range of $f(x) = 3 + 5 \sin x$.

$$-1 \leq \sin x \leq +1 \rightarrow -5 \leq 5 \sin x \leq +5 \rightarrow$$

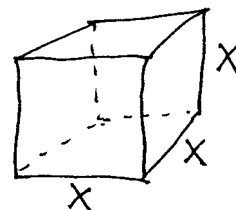
$$-2 \leq 3 + 5 \sin x \leq 8 \text{ so}$$

Range: $-2 \leq y \leq 8$

- 3.) Consider a three-dimensional cube with side length x .

- a.) Write the volume V of the cube as a function of x .

$$V = x^3$$



- b.) Write the surface area S of the cube as a function of x .

$$S = 6x^2$$

- c.) Write the surface area S of the cube as a function of the volume V .

$$V = x^3 \rightarrow x = V^{1/3} \rightarrow$$

$$S = 6x^2 = 6(V^{1/3})^2 = 6V^{2/3}$$

The following EXTRA CREDIT PROBLEM is worth points. This problem is OPTIONAL.

1.) Determine the following limit : $\lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 9x}) = \infty - \infty$

$$\lim_{x \rightarrow \infty} \frac{(x - \sqrt{x^2 + 9x})(x + \sqrt{x^2 + 9x})}{(x + \sqrt{x^2 + 9x})}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 9x)}{x + \sqrt{x^2 + 9x}}$$

$$= \lim_{x \rightarrow \infty} \frac{-9x}{x + \sqrt{x^2 + 9x}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow \infty} \frac{-9}{1 + \sqrt{x^2 + 9x} \cdot \sqrt{\frac{1}{x^2}}}$$

$$= \lim_{x \rightarrow \infty} \frac{-9}{1 + \sqrt{1 + \frac{9}{x}}} =$$

$$= \frac{-9}{1 + \sqrt{1 + 0}}$$

$$= -\frac{9}{2}$$