1. (a) State the PRECISE definition of the limit for

\[ \lim_{x \to a} f(x) = L \]

- Given any number \( \varepsilon > 0 \), there exists another number \( \delta > 0 \) such that if \( |x-a| < \delta \), then \( |f(x) - L| < \varepsilon \)

o r

- \( \forall \varepsilon > 0, \exists \delta > 0 \) s.t. \( |x-a| < \delta \implies |f(x) - L| < \varepsilon \)

(b) Give an \( \varepsilon, \delta \)-proof for the following limit. This is a writing exercise.

\[ \lim_{x \to 1} \frac{3}{x+7} = \frac{1}{2} \]

\( \exists \varepsilon > 0 \), need to find \( \delta > 0 \) s.t.

\[ |x - (1)| = |x+1| < \delta \implies |f(x) - \frac{1}{2}| < \varepsilon \]

Begin w/ \( |f(x) - \frac{1}{2}| < \varepsilon \) & solve for \( |x+1| < \delta \)

\[ |f(x) - \frac{1}{2}| < \varepsilon \implies |\frac{3}{x+7} - \frac{1}{2}| < \varepsilon \implies |\frac{6-x-7}{2(x+7)}| < \varepsilon \]

\[ \implies \frac{1}{2} \left| \frac{-(x+1)}{x+7} \right| < \varepsilon \implies \frac{|x+1|}{2|x+7|} < \varepsilon \]

Want to remove \( |x+7| \)

Assume \( \delta \leq 1 \)

\[ \frac{a-\delta}{2} < \frac{a}{-1} < \frac{a+\delta}{0} \]

\( \implies -2 < x < 0 \implies 5 < |x+7| < 7 \implies \frac{1}{7} < \frac{1}{|x+7|} < \frac{1}{5} \)

Then \( \frac{|x+1|}{2|x+7|} < \frac{|x+1|}{2 \cdot 5} < \varepsilon \implies |x+1| < 10\varepsilon \)

Choose \( \delta = \min \{10\varepsilon, 1\} \)

Hence, if \( |x+1| < \delta \), it follows that \( |f(x) - \frac{1}{2}| < \varepsilon \)
2. Determine the following limits

(a) \[ \lim_{x \to 0} \frac{(e^x - 1)^2}{\sin(x^2)} = \frac{0}{0} \]
\[ = \lim_{x \to 0} \frac{e^x \cdot e^x}{\sin(x^2)} \cdot \frac{2x \cdot e^x - e^x}{2x} \cdot \frac{1}{\cos(x^2)} \cdot \frac{1}{2x} \cdot \frac{1}{\cos(x^2)} \]
\[ = \frac{1}{1} = 1 \]

(b) \[ \lim_{x \to 1^-} \frac{e^{2x} - e^x}{\cos(x^2)(2x)} = \frac{0}{0} \]
\[ = \lim_{x \to 1^-} \frac{e^{2x} \cdot 2}{\cos(x^2) \cdot 2x} \cdot \frac{1}{x} \cdot \frac{1}{\cos(x^2)} \]
\[ = -\sqrt{2} \]

(c) \[ \lim_{x \to \infty} (3 + 2x)^{\frac{1}{x}} = \frac{\infty^0}{0} \]
\[ = \lim_{x \to \infty} e^{\ln(3 + 2x)^{\frac{1}{x}}} \]
\[ = e^{\lim_{x \to \infty} \frac{1}{x} \ln(3 + 2x)} \]
\[ = e^{\lim_{x \to \infty} \frac{\ln(3 + 2x)}{x}} \]
\[ = e^0 = 1 \]

3. (8 pts) Consider the following function

\[ f(x) = \begin{cases} 
Ax + B & \text{if } x < 0 \\
12 & \text{if } 0 \leq x \leq 2 \\
Bx^2 - A & \text{if } x > 2 
\end{cases} \]

Using limits, determine all constants \( A \) and \( B \) such that the function is continuous for all values of \( x \).

Want \[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \]
\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \]

\[ A(0) + B = 12 \]
\[ \Rightarrow B = 12 \]

\[ 4B - A = 12 \]
\[ \Rightarrow A = 4B - 12 = 4(12) - 12 \]
\[ \Rightarrow A = 36 \]
4. Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

(a) \( y = \arcsin (2^x + \log x) \)

\[
y' = \frac{1}{\sqrt{1 - (2^x + \log x)^2}} \left[ 2^x \ln x + \frac{1}{x} \frac{1}{\ln 10} \right]
\]

(b) \( f(t) = \cot \left( \frac{\sin t}{t} \right) \)

\[
f'(t) = -\csc^2 \left( \frac{\sin t}{t} \right) \left[ \frac{\cos t}{t^2} + \frac{1}{t} \sin t \right]
\]

(c) \( y = (2x)^{x+3} \)

\[
\ln y = \ln (2x)^{x+3} = (x+3) \ln 2x \quad \Rightarrow \quad \frac{y'}{y} = \ln 2x + \frac{x+3}{2x} \quad \Rightarrow \\
\Rightarrow y' = y \left[ \ln 2x + \frac{x+3}{x} \right] = (2x)^{x+3} \left[ \ln 2x + \frac{x+3}{x} \right]
\]

5. Use differentials to estimate the value of \( \sqrt[3]{26.5} \)

Let \( f(x) = \sqrt[3]{x} \) with \( x : 27 \to 26.5 \Rightarrow \Delta x = \frac{-1}{2} \)

\[
f'(x) = \frac{1}{3} x^{-\frac{2}{3}}
\]

\[
df = f'(27) \cdot \Delta x = \frac{1}{3 \cdot 27} \cdot \left( -\frac{1}{2} \right) = -\frac{1}{54}
\]

\[
\sqrt[3]{26.5} = f(x+\Delta x) \approx f(x) + df = 3 + \frac{1}{54}
\]

\[
\Rightarrow \sqrt[3]{26.5} \approx 2 + \frac{53}{54}
\]
6. (a) State the definition of the derivative.

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}
\]

(b) Use the definition of the derivative to differentiate the function

\[f(x) = \sqrt{2x + 1}\]

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{2(x+h) + 1} - \sqrt{2x + 1}}{h} \cdot \frac{\sqrt{2(x+h) + 1} + \sqrt{2x + 1}}{\sqrt{2(x+h) + 1} + \sqrt{2x + 1}}
\]

\[
= \lim_{h \to 0} \frac{2x + 2h + 1 - 2x - 1}{h(\sqrt{2(x+h) + 1} + \sqrt{2x + 1})} = \lim_{h \to 0} \frac{2h}{h(\sqrt{2(x+h) + 1} + \sqrt{2x + 1})}
\]

\[
= \frac{2}{2\sqrt{2x + 1}} = \frac{1}{\sqrt{2x + 1}}
\]

7. Use Newton's Method to approximate the solution to \(x^3 + x - 3 = 0\). Let \(x_0 = 0\) and perform ONLY TWO iterations of Newton's method (i.e. find \(x_2\)).

Let \(f(x) = x^3 + x - 3\) \(\Rightarrow f'(x) = 3x^2 + 1\)

\[x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n - 3}{3x_n^2 + 1}
\]

\[= \frac{3x_n^3 + x_n - x_n^3 - x_n + 3}{3x_n^2 + 1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}
\]

\[\Rightarrow x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}
\]

\[x_0 = 0 \Rightarrow x_1 = \frac{3}{1} = 3 \Rightarrow x_2 = \frac{2.27^3 + 3}{2.27 + 1} = \frac{57}{28}
\]
8. Use the Intermediate Value Theorem to show that the equation $x^3 + x = \sqrt{x+4}$ is solvable. This is a writing exercise.

$$x^3 + x = \sqrt{x+4} \quad \iff \quad x^3 + x - \sqrt{x+4} = 0$$

Let $f(x) = x^3 + x - \sqrt{x+4}$ & $m = 0$

$f$ is cont. (poly. + sqrt. is cont.)

Since $f(0) = -2 < 0$ & $f(5) = 5^3 + 5 - 3 > 0$

Choose interval $[0, 5]$ since $f(0) < m = 0 < f(5)$

By IMVT, there exists at least 1 $c$ b/w 0 & 5 s.t. $f(c) = 0 \iff c^3 + c = \sqrt{c+4}$

9. Find the slope and concavity of the graph $xy + y^2 = 3x + 1$ at the point $(0, -1)$

$$xy + y^2 = 3x + 1 \quad \iff \quad y + xy' + 2yy' = 3$$

$$\Rightarrow \quad (x + 2y)y' = 3 - y \quad \Rightarrow \quad y' = \frac{3 - y}{x + 2y} = \frac{y}{-2} = \boxed{-2 = \text{slope}}$$

$$y'' = \frac{-y'(x + 2y) - (3 - y)(1 + 2y')}{(x + 2y)^2}$$

$$= \frac{2(0 + 2(-1)) - (4)(1 - 4)}{4} = \frac{8}{4} = \boxed{2 = y''}$$

concave up
9. Use a differential to linearize the following function at the center point of $x = 5$

$$f(x) = \sqrt{x + 4}$$

$$f'(x) = \frac{1}{2\sqrt{x + 4}}$$

Let $x : 5 \rightarrow 5 + h \Rightarrow \Delta x = h$

$$df = f'(5) \Delta x = \frac{1}{6} h$$

$$\Rightarrow \sqrt{(5 + h) + 4} = f(x + \Delta x) \approx f(x) + df = f(5) + \frac{1}{6} h = 3 + \frac{1}{6} h$$

$$\Rightarrow \sqrt{h + 9} \approx 3 + \frac{1}{6} h$$

11.

(b) Determine if the following function satisfies the assumptions of the MVT on the given closed interval. If so, find all values of $c$ guaranteed by the conclusion of the MVT.

$$f(x) = \ln(x - 1) \text{ on } [2, 4]$$

$f$ is cont. on $[2, 4]$ (logs are cont. for $x > 0$).

$f'(x) = \frac{1}{x-1}$ is diff. on $(2, 4)$.

By MVT, there exists one $c$ b/w 2 and 4 s.t.

$$\frac{1}{c-1} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\ln 3 - \ln 1}{4 - 2} = \frac{\ln 3}{2}$$

$$\Rightarrow c - 1 = \frac{2}{\ln 3} \Rightarrow c = \frac{2}{\ln 3} + 1.$$
13. Find the point \((x, y)\) on the graph of \(y = \sqrt{x}\) which is nearest to the point \((4, 0)\)

**Goal:** Find pt. to min distance \(d\)

\[
d = \sqrt{(x-4)^2 + y^2}
\]

\[
\Rightarrow d(x) = \sqrt{(x-4)^2 + x}
\]

\[
\Rightarrow d'(x) = \frac{1}{2} \frac{1}{\sqrt{(x-4)^2 + x}} \cdot \left[2(x-4) + 1\right]
\]

\[
= \frac{1}{2} \frac{2x-7}{\sqrt{(x-4)^2 + x}} = 0 \quad x = \frac{7}{2}
\]

\[
\text{Pt. is} \quad \left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)
\]

14. Gifted White Rabbit has done a market analysis on pricing verses number of apps sold. This analysis shows that 400 apps are sold when the price is $2. For each $1 increase in price, 25 fewer apps are sold. What price per app will maximize total revenue?

**Goal:** Find price to max. revenue

Let \(x\) be \# of price increases

\[
\Rightarrow \text{Price} = 2 + x
\]

\[
\# \text{ sold} = 400 - 25x = 25(16-x)
\]

So \(\text{Revenue} = R(x) = 25(16-x)(2+x)\)

\[
\Rightarrow R'(x) = 25 \left[ 14 - 2x \right] = 0
\]

\[
x = 7
\]

\[
\text{Price} = 2 + 7 = \$9
\]

\[
\# \text{ sold} = 275
\]
Consider the following function with its corresponding first and second derivatives

\[ f(x) = (x - 1)^3(x - 5), \quad f'(x) = (x - 1)^2(4x - 16), \quad f''(x) = 4(x - 1)(3x - 9) \]

Determine where \( f \) is increasing, decreasing, concave up, and concave down. Identify all relative and absolute extrema, inflection points, vertical and horizontal asymptotes, and x- and y- intercepts. Neatly sketch the graph.

\[ f'(x) = 4(x - 1)^2(x - 4) = 0 \]

\[ f''(x) = 12(x - 1)(x - 3) = 0 \]

\( f \) is \( \uparrow \) for \( x > 4 \)
\( f \) is \( \downarrow \) for \( x < 4 \)
\( f \) is \( U \) for \( x < 1, x > 3 \)
\( f \) is \( \wedge \) for \( 1 < x < 3 \)
\( y \)-int.: \( x = 0 \Rightarrow y = 5 \)
\( x \)-int.: \( y = 0 \Rightarrow x = 1, 5 \)
9.) Car B is 34 miles directly east of car A and begins moving west at 90 mph. At the same moment car A begins moving north at 60 mph.

a.) At what rate is the distance between the cars changing after \( t = \frac{1}{5} \) hr.? 

\[
\frac{dy}{dt} = 60 \text{ mph}, \quad \frac{dx}{dt} = -90 \text{ mph},
\]

\[
\frac{dz}{dt} \quad \text{when} \quad t = \frac{1}{5} \text{ hr} \Rightarrow x = 12 \text{ mi}, \quad y = 16 \text{ mi}.
\]

\[
x^2 + y^2 = z^2 \rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}
\]

\[
\rightarrow (16)(-90) + (12)(60) = (2z) \frac{dz}{dt}
\]

\[
\rightarrow \frac{dz}{dt} = -36 \text{ mph}
\]

b.) What is the minimum distance between the cars and at what time \( t \) does the minimum distance occur?

Let \( t \) be time, minimize distance

\[
z = \sqrt{(60t)^2 + (34 - 90t)^2} \rightarrow
\]

\[
\frac{dz}{dt} = \frac{1}{2} \left( \frac{1}{z} \right)^{\frac{1}{2}} \left[ 2(60t)(60) + 2(34 - 90t)(-90) \right] = 0 \rightarrow
\]

\[
360t - 306 + 810t = 0 \rightarrow t \approx 0.26 \text{ hr.}
\]

and min. \( z \approx 18.86 \text{ mi.} \)
The following EXTRA CREDIT PROBLEM is worth This problem is OPTIONAL.

1.) You are standing at point A on the edge of a river 1 mile wide. You are to get to point B, which is 4 miles from the point directly across the river from you. You can paddle a canoe in the water at a speed of 10 miles per hour and you can ride a bicycle on land at a speed of 15 miles per hour. Determine x so that the time it takes to go from point A to point B is a minimum and determine the minimum time.

\[ D = RT \text{ so } T = \frac{D}{R}; \text{ minimize time} \]

\[ T = T_{\text{water}} + T_{\text{land}} \]

\[ T = \frac{\sqrt{x^2 + 1}}{10} + \frac{4-x}{15} \]

\[ T' = \frac{1}{10} \cdot \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \cdot 2x - \frac{1}{15} \]

\[ = \frac{x}{10\sqrt{x^2 + 1}} - \frac{1}{15} = 0 \rightarrow \frac{x}{10\sqrt{x^2 + 1}} = \frac{1}{15} \]

\[ 15x = 10\sqrt{x^2 + 1} \rightarrow 225x^2 = 100(x^2 + 1) \rightarrow 225x^2 = 100x^2 + 100 \rightarrow 125x^2 = 100 \rightarrow x^2 = \frac{160}{125} \rightarrow x = \sqrt{\frac{160}{125}} \approx 0.894 \text{ mi} \]

\[ x = \sqrt{\frac{160}{125}} \text{ mi, min } T \approx 0.34 \text{ hr.} \]