1.

(a) State the PRECISE definition of the limit for

$$\lim_{x \to a} f(x) = L$$

- Given any number $\varepsilon > 0$, there exists another number $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$

or

- $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

(b) Give an ε, δ-proof for the following limit. This is a writing exercise.

$$\lim_{x \to -1} \frac{3}{x+7} = \frac{1}{2}$$

$\S$ $\varepsilon > 0$ is given. Need to find $\delta > 0$ s.t.

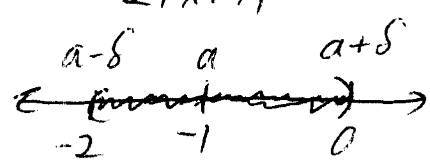
$|x - (-1)| = |x+1| < \delta \Rightarrow |f(x) - \frac{1}{2}| < \varepsilon$

Begin w/ $|f(x) - \frac{1}{2}| < \varepsilon$ & solve for $|x+1| < \delta$

$|f(x) - \frac{1}{2}| < \varepsilon \Leftrightarrow \left|\frac{3}{x+7} - \frac{1}{2}\right| < \varepsilon \Leftrightarrow \left|\frac{6-x-7}{2(x+7)}\right| < \varepsilon$

$\Leftrightarrow \frac{1}{2}\left|\frac{-(x+1)}{x+7}\right| < \varepsilon \Leftrightarrow \frac{|x+1|}{2|x+7|} < \varepsilon$. Want to remove $|x+7|$

Assume $\delta \leq 1$



$\Rightarrow -2 < x < 0 \Rightarrow 5 < |x+7| < 7 \Rightarrow \frac{1}{7} < \frac{1}{|x+7|} < \frac{1}{5}$

Then $\frac{|x+1|}{2|x+7|} < \frac{|x+1|}{2 \cdot 5} < \varepsilon \Leftrightarrow |x+1| < 10\varepsilon$

Choose $\delta = \min\{10\varepsilon, 1\}$

Hence, if $|x+1| < \delta$, it follows that $\left|f(x) - \frac{1}{2}\right| < \varepsilon$ $\blacksquare$

2. Determine the following limits

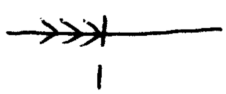
(a) $\lim\limits_{x\to 0} \dfrac{(e^x-1)^2}{\sin(x^2)} = \text{``}\dfrac{0}{0}\text{''}$

$\underset{L'H}{=} \lim\limits_{x\to 0} \dfrac{\cancel{2}(e^x-1)e^x}{\cos(x^2)(\cancel{2}x)} \text{ ``}\dfrac{0}{0}\text{''} \underset{L'H}{=} \lim\limits_{x\to 0} \dfrac{\cancel{e^x\cdot e^x}\ 2e^{2x}-e^x}{-\sin(x^2)\cdot 2x\cdot x+\cos(x^2)\cdot 1} = \dfrac{1}{1} = 1$

$\parallel$

$\dfrac{e^{2x}-e^x}{\cos(x^2)(2x)}$

(b) $\lim\limits_{x\to 1^-} \dfrac{\sqrt{2x}(x-1)}{|x-1|} \text{ ``}\dfrac{0}{0}\text{''} = \lim\limits_{x\to 1^-} \dfrac{\sqrt{2x}\,\cancel{(x-1)}}{-\cancel{(x-1)}} = -\sqrt{2}$



(c) $\lim\limits_{x\to\infty} (3+2x)^{\frac{1}{x}} = \text{``}\infty^0\text{''}$

$= \lim\limits_{x\to\infty} e^{\ln(3+2x)^{1/x}} = \lim\limits_{x\to\infty} e^{\frac{1}{x}\ln(3+2x)} = e^{\lim\limits_{x\to\infty} \frac{\ln(3+2x)}{x}} \text{ ``}\dfrac{\infty}{\infty}\text{''}$

$\underset{L'H}{=} e^{\lim\limits_{x\to\infty} \frac{\frac{1}{3+2x}\cdot 2}{1}} = e^0 = 1$

3. (*8 pts*) Consider the following function

$$f(x) = \begin{cases} Ax+B & \text{if } x<0 \\ 12 & \text{if } 0\le x\le 2 \\ Bx^2-A & \text{if } x>2 \end{cases}$$

Using limits, determine all constants A and B such that the function is continuous for all values of x.

Want $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$ & $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$

$A(0)+B = 12$

$\Rightarrow \boxed{B=12}$

$4B - A = 12$

$\Rightarrow A = 4B-12 = 4(12)-12$

$\Rightarrow \boxed{A=36}$

4. Differentiate each of the following functions. DO NOT SIMPLIFY ANSWERS.

(a) $y = \arcsin(2^x + \log x)$

$$y' = \frac{1}{\sqrt{1-(2^x+\log x)^2}}\left[2^x \ln x + \frac{1}{x}\frac{1}{\ln 10}\right]$$

(b) $f(t) = \cot\left(\frac{\sin t}{t}\right)$

$$f'(t) = -\csc^2\left(\frac{\sin t}{t}\right)\left[\frac{\cos t \cdot t - 1 \sin t}{t^2}\right]$$

(c) $y = (2x)^{x+3}$

$$\ln y = \ln (2x)^{x+3} = (x+3)\ln 2x \overset{D}{\Rightarrow} \frac{y'}{y} = \ln 2x + \frac{x+3}{\cancel{2}x}\cdot\cancel{2}$$

$$\Rightarrow y' = y\left[\ln 2x + \frac{x+3}{x}\right] = (2x)^{x+3}\left[\ln 2x + \frac{x+3}{x}\right]$$

5. Use differentials to estimate the value of $\sqrt[3]{26.5}$

Let $f(x) = \sqrt[3]{x}$ $\qquad \overset{x}{x: 27} \to \overset{x+\Delta x}{26.5} \Rightarrow \Delta x = -\frac{1}{2}$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$df = f'(27)\cdot\Delta x = \frac{1}{3\cdot 9}\cdot\left(-\frac{1}{2}\right) = -\frac{1}{54}$$

$$\sqrt{26.5} = f(x+\Delta x) \simeq f(x) + df = 3 - \frac{1}{54}$$

$$\Rightarrow \sqrt[3]{26.5} \simeq 2\frac{53}{54}$$

6.

(a) State the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Use the definition of the derivative to differentiate the function

$$f(x) = \sqrt{2x+1}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2(x+h)+1} + \sqrt{2x+1}}{\sqrt{2(x+h)+1} + \sqrt{2x+1}}$$

$$= \lim_{h \to 0} \frac{\cancel{2x} + 2h \cancel{+1} \cancel{-2x} \cancel{-1}}{h(\sqrt{2(x+h)+1} + \sqrt{2x+1})} = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{2(x+h)+1} + \sqrt{2x+1})}$$

$$= \frac{\cancel{2}}{\cancel{2}\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}$$

7. Use Newton's Method to approximate the solution to $x^3 + x - 3 = 0$. Let $x_0 = 0$ and perform ONLY TWO iterations of Newton's method (i.e. find x_2).

$$\text{Let } f(x) = x^3 + x - 3 \quad \Rightarrow f'(x) = 3x + 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n - 3}{3x_n^2 + 1}$$

$$= \frac{3x_n^3 + \cancel{x_n} - x_n^3 - \cancel{x_n} + 3}{3x_n^2 + 1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}$$

$$\Rightarrow x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}$$

$$x_0 = 0 \quad \Rightarrow \quad x_1 = \frac{3}{1} = 3 \quad \Rightarrow \quad x_2 = \frac{2 \cdot 27 + 3}{27 + 1} = \frac{57}{28}$$

8. Use the Intermediate Value Theorem to show that the equation $x^3 + x = \sqrt{x+4}$ is solvable. This is a writing exercise.

$x^3 + x = \sqrt{x+4} \iff x^3 + x - \sqrt{x+4} = 0$

Let $f(x) = x^3 + x - \sqrt{x+4}$ & $m = 0$

f is cont. (poly. + sqrt. is cont.)

Since $f(0) = -2 < 0$ & $f(5) = 5^3 + 5 - 3 > 0$

Choose interval $[0,5]$ since $f(0) < m = 0 < f(5)$

By IMVT, there exists @ least 1 c b/w 0 & 5

s.t. $f(c) = 0 \iff c^3 + c = \sqrt{c+4}$

9. Find the slope and concavity of the graph $xy + y^2 = 3x + 1$ at the point $(0, -1)$

$xy + y^2 = 3x + 1 \xRightarrow{D_x} y + xy' + 2yy' = 3$

$\Rightarrow (x+2y)y' = 3 - y \Rightarrow y' = \frac{3-y}{x+2y} = \frac{4}{-2} = \boxed{-2 = \text{slope}}$

$$y'' = \frac{-y'(x+2y) - (3-y)(1+2y')}{(x+2y)^2}$$

$$= \frac{2(0+2(-1)) - (4)(1-4)}{4} = \frac{8}{4} = \boxed{\begin{array}{l} 2 = y'' \\ \text{concave up} \end{array}}$$

9. Use a differential to linearize the following function at the center point of $x = 5$

$$f(x) = \sqrt{x+4}$$

$f'(x) = \frac{1}{2\sqrt{x+4}}$

Let $x : \overset{x}{5} \to \overset{x+\Delta x}{5+h} \Rightarrow \Delta x = h$

$df = f'(5)\Delta x = \frac{1}{6} h$

$\Rightarrow \sqrt{(5+h)+4} = f(x+\Delta x) \approx f(x) + df = f(5) + \frac{1}{6}h = 3 + \frac{1}{6}h$

$\Rightarrow \sqrt{h+9} \approx 3 + \frac{1}{6}h$

11.

(b) Determine if the following function satisfies the assumptions of the MVT on the given closed interval. If so, find all values of c guaranteed by the conclusion of the MVT.

$$f(x) = \ln(x-1) \text{ on } [2,4]$$

f is cont. on $[2,4]$ (logs are cont. for $x > 0$).

$f'(x) = \frac{1}{x-1}$ is diff. on $(2,4)$.

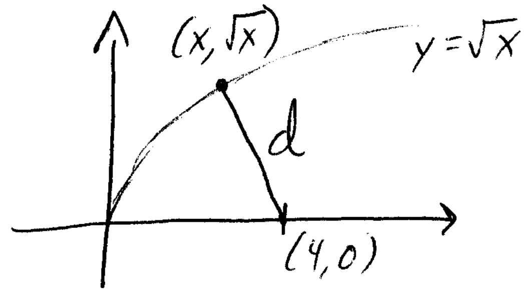
By MVT, there exists one c b/w 2 & 4 s.t.

$\frac{1}{c-1} = f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\ln 3 - \overset{0}{\cancel{\ln 1}}}{4-2} = \frac{\ln 3}{2}$

$\Rightarrow c - 1 = \frac{2}{\ln 3} \Rightarrow c = \frac{2}{\ln 3} + 1$.

13. Find the point (x, y) on the graph of $y = \sqrt{x}$ which is nearest to the point $(4, 0)$

Goal: Find pt. to min distance d

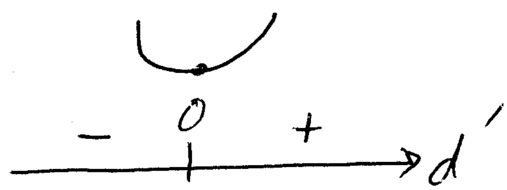


$$d = \sqrt{(x-4)^2 + y^2}$$

$$\Rightarrow d(x) = \sqrt{(x-4)^2 + x}$$

$$\Rightarrow d'(x) = \frac{1}{2} \frac{1}{\sqrt{(x-4)^2+x}} \cdot [2(x-4)+1]$$

$$= \frac{1}{2} \frac{(2x-7)}{\sqrt{(x-4)^2+x}} = 0 \qquad x = 7/2$$



$x = 7/2$, $y = \sqrt{7/2}$ } min

$$\boxed{\text{Pt. is } \left(\tfrac{7}{2}, \sqrt{\tfrac{7}{2}}\right)}$$

14. Gifted White Rabbit has done a market analysis on pricing verses number of apps sold. This analysis shows that 400 apps are sold when the price is \$2. For each \$1 increase in price, 25 fewer apps are sold. What price per app will maximize total revenue?

Goal: Find price to max. revenue

Let x be # of price increases

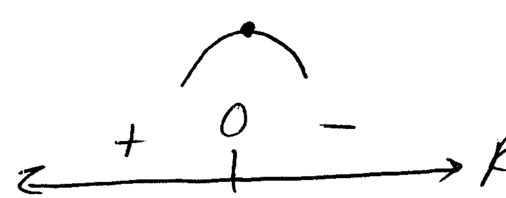
$$\Rightarrow \begin{aligned} \text{Price} &= 2 + x \\ \#\text{ sold} &= 400 - 25x = 25(16-x) \end{aligned}$$

So Revenue $= R(x) = 25(16-x)(2+x)$

$$\Rightarrow R'(x) = 25[(-1)(2+x) + (16-x)1]$$

$$= 25[14 - 2x] = 0$$

$$x = 7$$



$x = 7$, $R = 2{,}475$ } max

$$\boxed{\text{Price} = 2 + 7 = \$9}$$

sold = 275

15. Consider the following function with its corresponding first and second derivatives

$$f(x) = (x-1)^3(x-5),\ f'(x) = (x-1)^2(4x-16),\ f''(x) = 4(x-1)(3x-9)$$

Determine where f is increasing, decreasing, concave up, and concave down. Identify all relative and absolute extrema, inflection points, vertical and horizontal asymptotes, and x- and y- intercepts. Neatly sketch the graph.

$f'(x) = 4(x-1)^2(x-4) = 0$

$f'(x)$: $-$ 0 $-$ 0 $+$

? $\begin{cases} x=1 \\ y=0 \end{cases}$ $\quad$ $\left.\begin{matrix} x=4 \\ y=-27 \end{matrix}\right\}$ abs min

$f''(x) = 12(x-1)(x-3) = 0$

$f''(x)$: $+$ 0 $-$ 0 $+$

Inf. pt. $\begin{cases} x=1 \\ y=0 \end{cases}$ $\quad$ $\left.\begin{matrix} x=3 \\ y=-16 \end{matrix}\right\}$ Inf. pt.

f is $\uparrow$ for $x > 4$

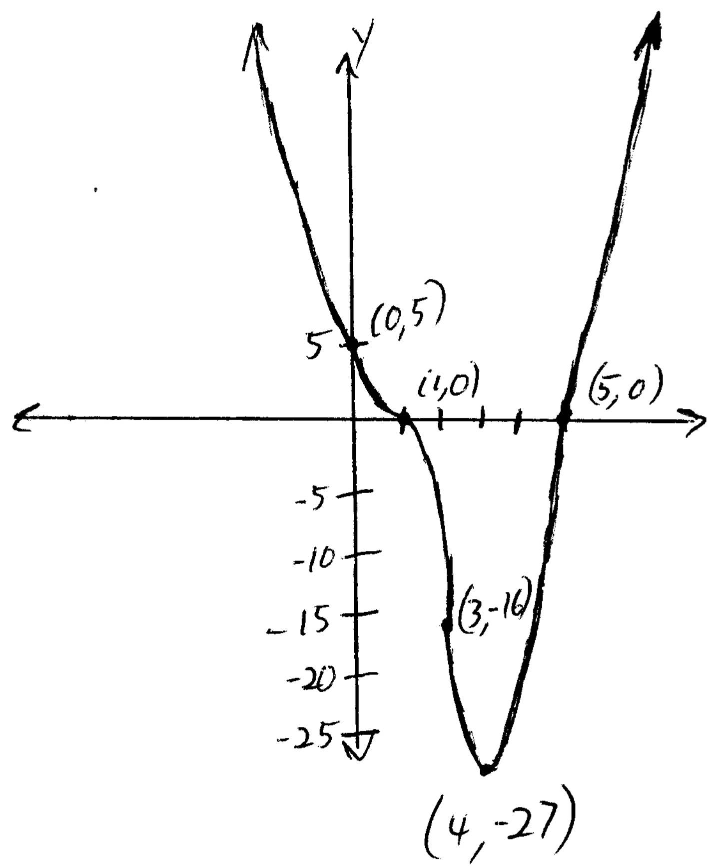
f is $\downarrow$ for $x < 4$

f is $\cup$ for $x<1, x>3$

f is $\cap$ for $1<x<3$

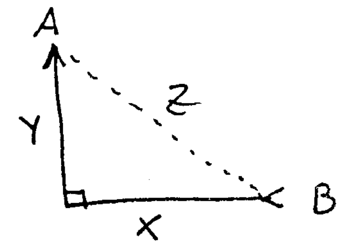
y-int: $x=0 \Rightarrow y=5$

x-int: $y=0 \Rightarrow x=1,5$

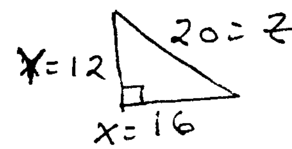


9.) Car B is 34 miles directly east of car A and begins moving west at 90 mph. At the same moment car A begins moving north at 60 mph.

a.) At what rate is the distance between the cars changing after $t = \frac{1}{5}$ hr. ?



$\frac{dY}{dt} = 60$ mph, $\frac{dX}{dt} = -90$ mph, find $\frac{dZ}{dt}$ when $t = \frac{1}{5}$ hr $\rightarrow X = 12$ mi, $Y = 16$ mi. :



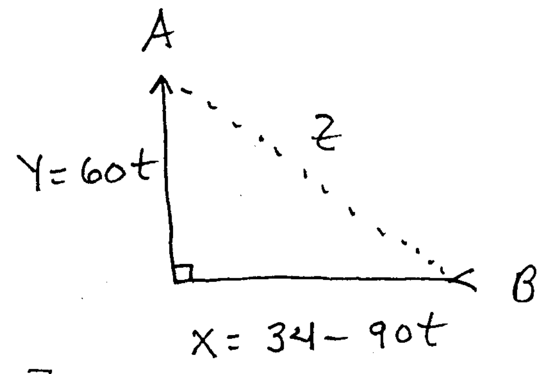
$$X^2 + Y^2 = Z^2 \rightarrow \cancel{2}X\frac{dX}{dt} + \cancel{2}Y\frac{dY}{dt} = \cancel{2}Z\frac{dZ}{dt}$$

$$\rightarrow (16)(-9\cancel{0}) + (12)(6\cancel{0}) = (2\cancel{0})\frac{dZ}{dt}$$

$$\rightarrow \frac{dZ}{dt} = -36 \text{ mph}$$

b.) What is the minimum distance between the cars and at what time t does the minimum distance occur ?

Let t be time, minimize distance



$$Z = \sqrt{(60t)^2 + (34-90t)^2} \rightarrow$$

$$\frac{dZ}{dt} = \frac{1}{2}(\sim)^{-\frac{1}{2}} \cdot [\cancel{2}(60t)6\cancel{0} + \cancel{2}(34-90t)\cdot(-9\cancel{0})] = 0 \rightarrow$$

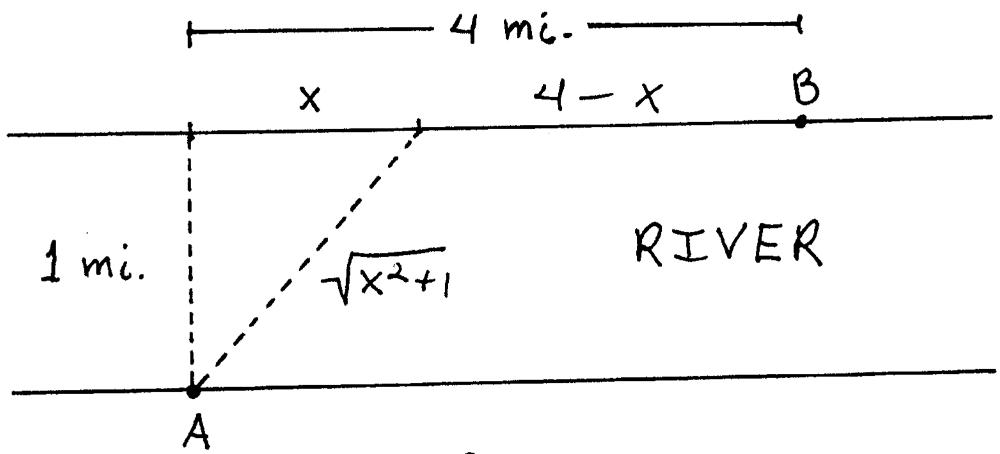
$$360t - 306 + 810t = 0 \rightarrow t \approx 0.26 \text{ hrs.}$$

$$\begin{array}{ccc} - & 0 & + \\ \hline & t=0.26 & \end{array} \; Z'$$

and min. $Z \approx 18.86$ mi.

The following EXTRA CREDIT PROBLEM is worth This problem is OPTIONAL.

1.) You are standing at point A on the edge of a river 1 mile wide. You are to get to point B, which is 4 miles from the point directly across the river from you. You can paddle a canoe in the water at a speed of 10 miles per hour and you can ride a bicycle on land at a speed of 15 miles per hour. Determine x so that the time it takes to go from point A to point B is a minimum and determine the minimum time.



$D = RT$ so $T = D/R$; minimize time

$T = T_{water} + T_{land} \rightarrow$

$T = \dfrac{\sqrt{x^2+1}}{10} + \dfrac{4-x}{15} \xrightarrow{D}$

$T' = \frac{1}{10} \cdot \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x - \frac{1}{15}$

$= \dfrac{x}{10\sqrt{x^2+1}} - \dfrac{1}{15} = 0 \rightarrow \dfrac{x}{10\sqrt{x^2+1}} = \dfrac{1}{15} \rightarrow$

$15x = 10\sqrt{x^2+1} \rightarrow 225x^2 = 100(x^2+1) \rightarrow$

$225x^2 = 100x^2 + 100 \rightarrow 125x^2 = 100 \rightarrow$

$x^2 = \dfrac{100}{125} \rightarrow x = \sqrt{\dfrac{100}{125}} \approx 0.894$ mi

$\underline{\quad - \quad 0 \quad + \quad}$ T'

$x = \sqrt{\dfrac{100}{125}}$ mi , min $T \approx 0.34$ hr.