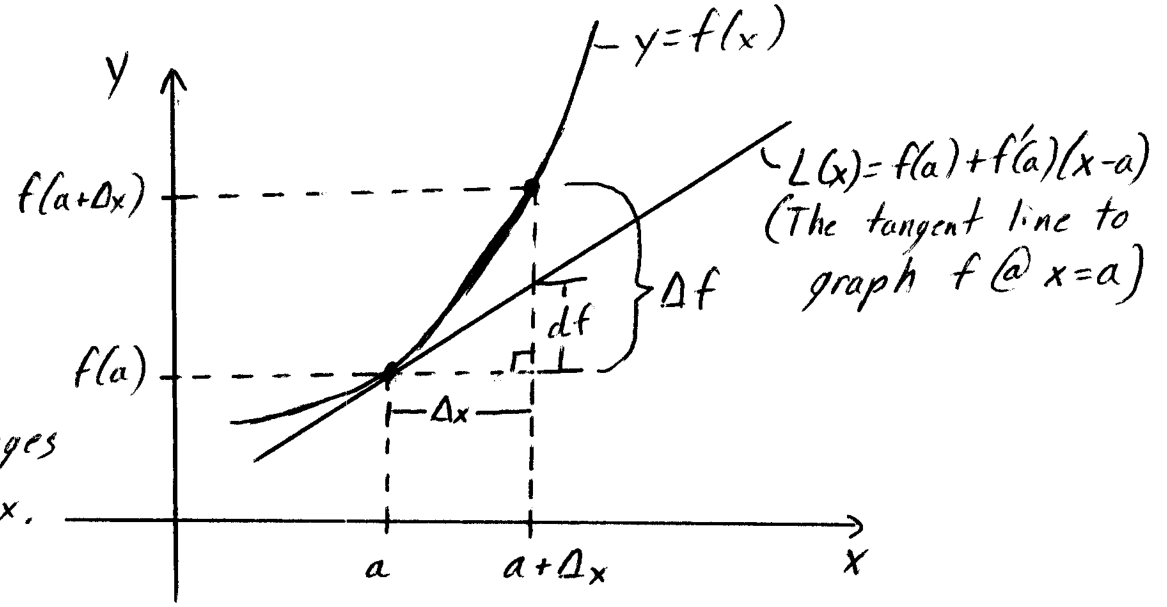
Math 21A
Vogler
The Differential

- Let Δx be the <u>change (error) in x</u>, & assume x changes from a to $a+\Delta x$. $(x: a \to a+\Delta x)$



- Define the <u>exact change</u> in f as

$$\boxed{\Delta f = \Delta y = f(a+\Delta x) - f(a)}$$

- Note: slope of line $L = \frac{\text{rise}}{\text{run}} \Rightarrow f'(a) = \frac{df}{\Delta x} \Rightarrow df = f'(a)\Delta x$
- Define the <u>differential</u> (approximate change) of f as

$$\boxed{df = f'(a)\Delta x}$$

- Fact: If Δx is 'small', then $\boxed{df \approx \Delta f}$

With this fact & the differential, we can approximate or simply functions by the following eqn for a line

$$\boxed{f(a+\Delta x) \approx f(a) + df = f(a) + f'(a)\Delta x}$$

A more convenient form for this line is obtained by letting $x = a+\Delta x$ ($\Leftrightarrow \Delta x = x-a$) and rewriting as

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$

This equation is called <u>the linearization of f @ $x=a$.</u>

Note: To use the linearization effectively, you must choose 'a' such that $f(a)$ & $f'(a)$ can be easily determined.

More examples using differentials

Example 1: For small h, show that $\sqrt{4+3h^2} \simeq 2 + \frac{3}{4}h^2$ using differentials.

Soln Let $f(x) = \sqrt{x}$ & assume that $x: 4 \to 4+3h^2$

$\Rightarrow \Delta x = 3h^2$ & $f'(x) = \frac{1}{2\sqrt{x}}$. Since Δx is small (b/c h is small)

$$\Delta f \simeq df \Rightarrow f(4+3h^2) - f(4) \simeq f'(4) \cdot \Delta x$$

$$\Rightarrow \sqrt{4+3h^2} - \sqrt{4} \simeq \frac{1}{2\sqrt{4}} \cdot 3h^2 \Rightarrow \sqrt{4+3h^2} - 2 \simeq \frac{3}{4}h^2$$

$$\Rightarrow \sqrt{4+3h^2} \simeq 2 + \frac{3}{4}h^2$$

Example 2: If the radius of a circle is measured w/ an absolute percentage error of @ most 3%, use differentials to estimate the maximum absolute percentage error in computing the circle's

a) circumference b) area

Solution: Assume that $\frac{|\Delta r|}{r} \leq 3\%$

a) $C = 2\pi r \Rightarrow C' = 2\pi$, find $\frac{|\Delta C|}{C}$:

$$\frac{|\Delta C|}{C} \simeq \frac{|dC|}{C} = \frac{|C'\Delta r|}{C} = \frac{|2\pi \cdot \Delta r|}{2\pi r} = \frac{|\Delta r|}{r} \leq 3\%$$

b) $A = \pi r^2 \Rightarrow A' = 2\pi r$, find $\frac{|\Delta A|}{A}$

$$\frac{|\Delta A|}{A} \simeq \frac{|dA|}{A} = \frac{|A'\Delta r|}{A} = \frac{|2\pi r\, \Delta r|}{\pi r^2} = 2\,\frac{|\Delta r|}{r} \leq 2(3\%) = 6\%$$