Let $\Delta x$ be the change (error) in $x$, & assume $x$ changes from $a$ to $a+\Delta x$. 

Define the exact change in $f$ as: 

$$\Delta f = \Delta y = f(a+\Delta x) - f(a)$$

- Note: slope of line $L = \frac{\text{rise}}{\text{run}} \Rightarrow f'(a) = \frac{df}{\Delta x} \Rightarrow df = f'(a)\Delta x$
- Define the differential (approximate change) of $f$ as: 

$$df = f'(a)\Delta x$$

- Fact: If $\Delta x$ is 'small', then $df \approx \Delta f$

With this fact & the differential, we can approximate or simply functions by the following eqn for a line:

$$f(a+\Delta x) \approx f(a) + df = f(a) + f'(a)\Delta x$$

A more convenient form for this line is obtained by letting $x = a + \Delta x \ (\Rightarrow \ \Delta x = x - a)$ and rewriting as:

$$L(x) = f(a) + f'(a)(x-a)$$

This equation is called the linearization of $f(@x=a)$.

Note: To use the linearization effectively, you must choose 'a' such that $f(a) \& f'(a)$ can be easily determined.
More examples using differentials

Example 1: For small $h$, show that \( \sqrt{4 + 3h^2} \approx 2 + \frac{3}{4} h^2 \) using differentials.

\[ \text{Soln Let } f(x) = \sqrt{x} \quad \& \quad \text{assume that } x: 4 \to 4 + 3h^2 \]
\[ = \Delta x = 3h^2 \quad \& \quad f'(x) = \frac{1}{2\sqrt{x}}. \text{ Since } \Delta x \text{ is small } (\text{b/c } h \text{ is small}) \]
\[ \Delta f \approx df \quad \Rightarrow \quad f(4 + 3h^2) - f(4) \approx f'(4) \Delta x \]
\[ = \sqrt{4 + 3h^2} - \sqrt{4} \approx \frac{1}{2\sqrt{4}} \cdot 3h^2 = \sqrt{4 + 3h^2} - 2 \approx \frac{3}{4} h^2 \]
\[ = \sqrt{4 + 3h^2} \approx 2 + \frac{3}{4} h^2 \]

Example 2: If the radius of a circle is measured with an absolute percentage error of at most 3%, use differentials to estimate the maximum absolute percentage error in computing the circle's

a) circumference
b) area

Solution: Assume that \( \frac{|\Delta r|}{r} \leq 3\%

a) \( C = 2\pi r \Rightarrow C' = 2\pi \), find \( \frac{|\Delta C|}{C} \):
\[ \frac{|\Delta C|}{C} \approx \frac{|\Delta r|}{r} = \frac{1}{2\pi} \frac{|\Delta r|}{r} = \frac{1}{2\pi} \frac{|\Delta r|}{r} \leq 3\% \]

b) \( A = \pi r^2 \Rightarrow A' = 2\pi r \), find \( \frac{|\Delta A|}{A} \):
\[ \frac{|\Delta A|}{A} \approx \frac{|\Delta A|}{A} = \frac{1}{A} \frac{|\Delta r|}{r} = \frac{2\pi r}{\pi r^2} = 2 \frac{|\Delta r|}{r} \leq 2(3\%) = 6\% \]