Math 16A
Vogler
The Differential

- Let \( \Delta x \) be the change (error) in \( x \)

\& assume \( x \) changes from \( a \) to \( a + \Delta x \).

\( x : a \to a + \Delta x \)

- Define the exact change in \( f \) as

\[ \Delta f = \Delta y = f(a + \Delta x) - f(a) \]

- Note: slope of line \( L = \frac{\text{rise}}{\text{run}} \Rightarrow f'(a) = \frac{df}{\Delta x} \Rightarrow df = f'(a) \Delta x \)

- Define the differential (approximate change) of \( f \) as

\[ df = f'(a) \Delta x \]

- Fact: If \( \Delta x \) is 'small', then

\[ df \approx \Delta f \]

With this fact & the differential, we can approximate or simply functions by the following eqn for a line

\[ f(a + \Delta x) \approx f(a) + df = f(a) + f'(a) \Delta x \]

Note: To use this equation effectively, you must choose 'a' such that \( f(a) \& f'(a) \) can be easily determined.
Examples using differentials

Example 1: Use differentials to estimate \( \sqrt[3]{26.5} \).

Let \( f(x) = \sqrt[3]{x} \) and \( a = 27 \) \( (b/c \ 27 \approx 26.5 \) and \( \sqrt[3]{27} \) is computable \( ) \).

\( x: 27 \rightarrow 26.5 \Rightarrow \Delta x = \frac{1}{2} \); \( f'(x) = \frac{1}{3(\sqrt[3]{x})^2} \)

\( \Delta y = f(26.5) - f(27) = \sqrt[3]{26.5} - \sqrt[3]{27} = \sqrt[3]{26.5} - 3 \)

\( dy = f'(27) \cdot \Delta x = \frac{1}{3(\sqrt[3]{27})^2} \cdot (-\frac{1}{2}) = -\frac{1}{54} \)

\( \Delta y \approx dy \Rightarrow \sqrt[3]{26.5} - 3 \approx -\frac{1}{54} \)

\( \Rightarrow \sqrt[3]{26.5} \approx 3 - \frac{1}{54} = 2 \frac{53}{54} \)

Example 2: If the radius of a circle is measured w/ an absolute percentage error of at most 3%, use differentials to estimate the maximum absolute percentage error in computing the circle’s

a) circumference \quad b) area

Solution: Assume that \( \frac{|\Delta r|}{r} \leq 3\%

a) \( C = 2\pi r \Rightarrow C' = 2\pi \), find \( \frac{|\Delta C|}{C} \):

\( \frac{|\Delta C|}{C} \approx \frac{|dC|}{C} = \frac{1C' \Delta r}{C} = \frac{2\pi \Delta r}{2\pi r} = \frac{|\Delta r|}{r} \leq 3\% \)

b) \( A = \pi r^2 \Rightarrow A' = 2\pi r \), find \( \frac{|\Delta A|}{A} \):

\( \frac{|\Delta A|}{A} \approx \frac{|dA|}{A} = \frac{1A' \Delta r}{A} = \frac{2\pi r \Delta r}{\pi r^2} = 2 \frac{|\Delta r|}{r} \leq 2(3\%) = 6\% \)