Math 125B
Vogler

Chain Rule and Mean Value Theorem for Multidimensions

Thm (11.28) Chain Rule

Let $\vec{g}: U \to \mathbb{R}^m$ for an open set $U \subseteq \mathbb{R}^n$ and $\vec{f}: V \to \mathbb{R}^p$ for an open set $V \subseteq \mathbb{R}^m$ where $\vec{g}(U) \subseteq V$. If $\vec{g}$ is differentiable at $\vec{a} \in U$ and $\vec{f}$ is differentiable at $\vec{g}(\vec{a})$, then the composite function $f \circ g: U \to \mathbb{R}^p$ is differentiable at $\vec{a}$ and

$$D[\vec{f} \circ \vec{g}](\vec{a}) = D\vec{f}(\vec{g}(\vec{a})) \cdot D\vec{g}(\vec{a}). \quad (1)$$

- Note: By definition, we have $D\vec{f}(\vec{g}(\vec{a})): \mathbb{R}^m \to \mathbb{R}^p$ and $D\vec{g}(\vec{a}): \mathbb{R}^n \to \mathbb{R}^m$, so that $D\vec{f}(\vec{g}(\vec{a})) \cdot D\vec{g}(\vec{a}): \mathbb{R}^n \to \mathbb{R}^p$ is well-defined. With respect to the standard basis of vectors $\hat{e}_i$, we can express the RHS of (1) as multiplication of two matrices; in particular, if $F := f \circ g$, (1) becomes

$$D\vec{F}(\vec{a}) := \left[\frac{\partial F_i}{\partial x_k}\right]_{p \times n} = \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial y_1} & \cdots & \frac{\partial f_p}{\partial y_m} \end{pmatrix} \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

where $\frac{\partial f_i}{\partial y_j}$ are evaluated at $\vec{y} = \vec{g}(\vec{a})$ and $\frac{\partial g_j}{\partial x_k}$ are evaluated at $\vec{x} = \vec{a}$. Using matrix multiplication, we can compute

$$\frac{\partial F_i}{\partial x_k}(\vec{a}) = \sum_{j=1}^{m} \frac{\partial f_i}{\partial y_j}(\vec{g}(\vec{a})) \cdot \frac{\partial g_j}{\partial x_k}(\vec{a}) \quad \text{for } i = 1, \ldots, p;\ k = 1, \ldots, n.$$

Defn Let $\vec{a}, \vec{x} \in \mathbb{R}^n$. The <u>line segment from $\vec{a}$ to $\vec{x}$</u> is the set of points $L(\vec{a}; \vec{x}) := \{(1-t)\vec{a} + t\vec{x} : t \in [0,1]\}$.

We sat $\vec{c} \in \mathbb{R}^n$ is <u>between $\vec{a}$ and $\vec{x}$</u> if $\vec{c} \in L(\vec{a}, \vec{x})$, which is equivalent to $\exists t \in [0,1]$ s.t. $\vec{c} = (1-t)\vec{a} + t\vec{x}$.

Thm (11.30) Mean Value Theorem for Real Valued Functions

Let $V \subseteq \mathbb{R}^n$ be open and $f: V \to \mathbb{R}$ be differentiable on V. If $\vec{x}, \vec{a} \in V$ and $L(\vec{x}, \vec{a}) \subseteq V$, then $\exists \vec{c} \in L(\vec{x}, \vec{a})$ such that

$$f(\vec{x}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{x} - \vec{a}).$$

Thm (11.32) Mean Value Theorem for Vector Valued Functions

Let $V \subseteq \mathbb{R}^n$ be open and $\vec{f}: V \to \mathbb{R}^m$ be differentiable on V. If $\vec{x}, \vec{a} \in V$ and $L(\vec{x}; \vec{a}) \subseteq V$, then $\forall \vec{u} \in \mathbb{R}^m$, $\exists c \in L(\vec{x}, \vec{a})$ such that $\vec{u} \cdot [\vec{f}(\vec{x}) - \vec{f}(\vec{a})] = \vec{u} \cdot [D\vec{f}(\vec{c}) \cdot (\vec{x} - \vec{a})]$.

-Note: For both Mean Value Theorems, the equalities are both scalar equations and cannot be extended to vector equations.

Defn $E \subseteq \mathbb{R}^n$ is convex if $L(\vec{x}; \vec{a}) \subseteq E \;\; \forall \vec{x}, \vec{a} \in E$.

-Note: This basically means every line between any two points in E is completely contained in E.

Cor (11.34) Let $V \subseteq \mathbb{R}^n$ be open and $H \subseteq V$ be compact. Suppose $\vec{f}: V \to \mathbb{R}^m$ and $\vec{f} \in C'(V)$. If $E \subseteq H$ is convex, then there exists constant M such that

$$\|\vec{f}(\vec{x}) - \vec{f}(\vec{a})\| \leq M \|\vec{x} - \vec{a}\| \quad \forall \vec{x}, \vec{a} \in E. \quad (2)$$

-Note: (2) is the multidimensional version of Lipschitz continuity.

Cor (11.35) Let $V \subseteq \mathbb{R}^n$ be open and connected, and let $\vec{f}: V \to \mathbb{R}^m$ be differentiable on V. If $D\vec{f}(\vec{c}) = O$ (as a matrix) $\forall \vec{c} \in V$, then $\vec{f}$ is constant on V.