Math 125B
Vogler

Properties of the Derivative

Thm (11.20)

Let $\alpha \in \mathbb{R}$, $\vec{a} \in \mathbb{R}^n$, and suppose $\vec{f}, \vec{g} : \mathbb{R}^n \to \mathbb{R}^m$. If $\vec{f}$ and $\vec{g}$ are differentiable at $\vec{a}$, then $\vec{f}+\vec{g}$, $\alpha\vec{f}$, $\vec{f}\cdot\vec{g}$ are differentiable at $\vec{a}$ as well. Moreover, we have

$$D(\vec{f}+\vec{g})(\vec{a}) = D\vec{f}(\vec{a}) + D\vec{g}(\vec{a}), \quad \text{(Sum Rule)}$$

$$D(\alpha\vec{f})(\vec{a}) = \alpha\, D\vec{f}(\vec{a}), \quad \text{(Constant Multiple Rule)}$$

and

$$D(\vec{f}\cdot\vec{g})(\vec{a}) = \vec{g}(\vec{a})\, D\vec{f}(\vec{a}) + \vec{f}(\vec{a})\, D\vec{g}(\vec{a}). \quad \text{(Dot Product Rule)}$$

-Note: The first two rules make differentiation a linear operator on functions, compare with definition is Section 8.2.

Defn

Let $S \subseteq \mathbb{R}^m$ and $\vec{c} \in S$. A hyperplane Π with normal $\vec{n}$ is *tangent to S* if $\vec{c} \in \Pi$ and

$$\lim_{k\to\infty} \vec{n} \cdot \frac{\vec{c}_k - \vec{c}}{\|\vec{c}_k - \vec{c}\|} = 0$$

for all sequences $\{\vec{c}_k\}$ with $\vec{c}_k \in S - \{\vec{c}\}$ $\forall k \in \mathbb{N}$ and $\lim_{k\to\infty} \vec{c}_k = \vec{c}$.

-Note: This is just a generalization to higher dimensions of the following

In 2D ($m=2$): S is a curve and Π is a tangent line.

In 3D ($m=3$): S is a surface and Π is a tangent plane.

Thm (11.22)

Let $V \subseteq \mathbb{R}^n$ is open, $\vec{a} \in V$, and $f: V \to \mathbb{R}$.
If f is differentiable at $\vec{a}$, then the surface S defined by

$$S := \{(\vec{x}, z) \in \mathbb{R}^{n+1} : z = f(\vec{x}) \text{ and } \vec{x} \in V\}$$

has a tangent hyperplane at $(\vec{a}, f(\vec{a}))$ with normal

$$\vec{n} = (\nabla f(\vec{a}), -1) := (f_{x_1}(\vec{a}), f_{x_2}(\vec{a}), \dots, f_{x_n}(\vec{a}), -1).$$

An equation of this tangent hyperplane is

$$z = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}). \quad (*)$$

-Notes: 1) Recall, the gradient vector $\nabla f(\vec{a}) := (f_{x_1}(\vec{a}), \dots, f_{x_n}(\vec{a}))$ is an n-dimensional vector that points in the direction of steepest ascent of f in the input space.

2) This is a generalization of the following

In 2D ($m=2$): S is a curve, $(*)$ is equation of tangent line, & $\vec{n}$ is $\perp$ to this line.

In 3D ($m=3$): S is a surface, $(*)$ is the equation of the tangent plane, and $\vec{n}$ is $\perp$ to this plane.

Defn Let $f: \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\vec{a}$.
Define exact change in f as $\Delta f = f(\vec{a} + \Delta\vec{x}) - f(\vec{a})$ where $\Delta\vec{x} := (\Delta x_1, \Delta x_2, \dots, \Delta x_n)$. Define the differential of f as

$$df := \nabla f(\vec{a}) \cdot \Delta\vec{x} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\vec{a}) \Delta x_j.$$

Thm Let the above definition hold. Then

$$\lim_{\Delta\vec{x} \to \vec{0}} \frac{\Delta f - df}{\|\Delta\vec{x}\|} = 0 \quad \text{or} \quad \lim_{\Delta\vec{x} \to \vec{0}} \frac{\Delta f}{\|\Delta\vec{x}\|} = \lim_{\Delta\vec{x} \to 0} \frac{df}{\|\Delta\vec{x}\|}.$$

-Note: When $\Delta\vec{x}$ is sufficeintly small, we have $\Delta f \simeq df$, or the differential closely approximates the exact change.