Definition of Differentiability

Unless otherwise stated, we will assume \( V \subseteq \mathbb{R}^n \) is an open set, \( \vec{a} \in V \), and \( \vec{f} : V \rightarrow \mathbb{R}^m \).

**Defn**

\( \vec{f} \) is **differentiable** at \( \vec{a} \) if there exists \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that

\[
\lim_{\|h\| \to 0} \frac{\vec{f}(\vec{a}+h) - \vec{f}(\vec{a}) - \vec{T}(h)}{\|h\|} = \vec{0} \quad (*)
\]

\( \vec{f} \) is differentiable on a set \( E \subseteq V \) if \( \vec{f} \) is differentiable at every pt. in \( E \).

- **Note**: When \( m=n=1 \), \( \vec{f} : \mathbb{R} \rightarrow \mathbb{R} \) and being differentiable \((*)\) is equivalent to the limit definition of the derivative learned in Math 125A where \( T(h) = f'(\vec{a})h \).

**Thm (11.13)**

If \( \vec{f} \) is differentiable at \( \vec{a} \), then \( \vec{f} \) is continuous at \( \vec{a} \).

**Thm (11.14)**

If \( \vec{f} \) is differentiable, then all the first-order partial derivatives of \( \vec{f} \) exist at \( \vec{a} \). Also, the **total derivative** of \( \vec{f} \) at \( \vec{a} \), denoted by \( \vec{D} \vec{f}(\vec{a}) \), is unique and is related to the partial derivatives by

\[
\vec{D} \vec{f}(\vec{a}) = \begin{bmatrix}
\frac{\partial f_1}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a})
\end{bmatrix}_{m \times n}
\]

- **Notes**: a) This gives us a convenient way to compute the linear operator \( T \) in \((*)\), which is \( T(h) = \vec{D} \vec{f}(\vec{a}) \cdot h \).

b) Some people refer to the above matrix as the **Jacobian Matrix**.
If we let the matrix \( B = Df(\bar{a}) \), then (*) can be rewritten as
\[
\lim_{h \to 0} \frac{f(\bar{a}+h) - f(\bar{a}) - Bh}{\|h\|} = \vec{0} \quad \text{or} \quad \lim_{h \to 0} \frac{\|f(\bar{a}+h) - f(\bar{a}) - Bh\|}{\|h\|} = 0
\]
and each of these forms can be used interchangeably.

**Thm (11.15)**
If all the first-order partial derivatives of \( \vec{f} \) exist in \( V \) and are continuous at \( \bar{a} \), then \( \vec{f} \) is differentiable at \( \bar{a} \).

Notes:
- a) The requirements are met if \( \vec{f} \in C'(V) \).
- b) This tends to be an easier way to show \( \vec{f} \) is differentiable over the definition.

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**Determining whether \( \vec{f} \) is differentiable at \( \bar{a} \)**

1. Compute all first order partials of \( \vec{f} \) at \( \bar{a} \). If one fails to exist, then \( \vec{f} \) is not differentiable.
2. If these partials all happen to be continuous at \( \bar{a} \), then \( \vec{f} \) is differentiable.
3. If one of the partials fails to be continuous, then you have to use the definition directly or one of the many forms (see c) above).