Math 125B
Vogler

Definition of Differentiability

- Unless otherwise stated, we will assume $V \subseteq \mathbb{R}^n$ is an open set, $\vec{a} \in V$, and $\vec{f}: V \to \mathbb{R}^m$.

Defn
$\vec{f}$ is *differentiable at $\vec{a}$* if there exists $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{\vec{h} \to \vec{0}} \frac{\vec{f}(\vec{a}+\vec{h}) - f(\vec{a}) - T(\vec{h})}{\|\vec{h}\|} = \vec{0} \quad (*).$$

$\vec{f}$ is *differentiable on a set* $E \subseteq V$ if $\vec{f}$ is differentiable at every pt. in E.

- Note: When $m=n=1$, $f: \mathbb{R} \to \mathbb{R}$ and being differentiable $(*)$ is equivalent to the limit definition of the derivative learned in Math 125A where $T(h) = f'(a)h$.

Thm (11.13)
If $\vec{f}$ is differentiable at $\vec{a}$, then $\vec{f}$ is continuous at $\vec{a}$.

Thm (11.14)
If $\vec{f}$ is differentiable, then all the first-order partial derivatives of $\vec{f}$ exist at $\vec{a}$. Also, the *total derivative of $\vec{f}$ at $\vec{a}$*, denoted by $D\vec{f}(\vec{a})$, is unique and is related to the partial derivatives by

$$D\vec{f}(\vec{a}) = \left[\frac{\partial f_i}{\partial x_j}(\vec{a})\right]_{m \times n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial f_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix}.$$

- Notes: a) This gives us a convenient way to compute the linear operator T in $(*)$, which is $T(\vec{h}) = D\vec{f}(\vec{a}) \cdot \vec{h}$.
b) Some people refer to the above matrix as the *Jacobian Matrix*.

c) If we let the matrix $B = D\vec{f}(\vec{a})$, then (*) can be rewritten as

$$\lim_{\vec{h}\to\vec{0}} \frac{\vec{f}(\vec{a}+\vec{h}) - f(\vec{a}) - B\vec{h}}{\|\vec{h}\|} = \vec{0} \quad \text{or} \quad \lim_{\vec{h}\to\vec{0}} \frac{\|\vec{f}(\vec{a}+\vec{h}) - \vec{f}(\vec{a}) - B\vec{h}\|}{\|\vec{h}\|} = 0$$

or $$\lim_{\vec{h}\to\vec{0}} \frac{\vec{f}(\vec{a}+\vec{h}) - \vec{f}(\vec{a}) - D\vec{f}(\vec{a})\vec{h}}{\|\vec{h}\|} = \vec{0},$$

and each of these forms can be used interchangeably.

<u>Thm (11.15)</u>

If all the first-order partial derivatives of $\vec{f}$ exist in V and are continuous at $\vec{a}$, then $\vec{f}$ is differentiable at $\vec{a}$.

-Notes: a) The requirements are met if $\vec{f} \in \mathcal{C}^1(V)$.

b) This tends to be an easier way to show $\vec{f}$ is differentiable over the definition.

<u>Determining whether $\vec{f}$ is differentiable at $\vec{a}$</u>

1) Compute all first order partials of $\vec{f}$ at $\vec{a}$. If one fails to exist, then $\vec{f}$ is not differentiable.

2) If these partials all happen to be continuous at $\vec{a}$, then $\vec{f}$ is differentiable.

3) If one of the partials fails to be continuous, then you have to use the definition directly or one of the many forms (see c) above).