Math 125B
Vogler

Fundamental Theorem of Calculus

Notation: Let I be interval $[a,b]$

- $\mathcal{R}(I)$ is the set of Riemann integrable functions on I.
- $C(I)$ is the set of continuous fuctions on I.
- $C'(I)$ is the set of continuously differentiable functions on I.

Thm (5.28) Fundamental Theorem of Calculus

1) If $f \in C(I)$ and $F(x) := \int_a^x f(t)\,dt$, then $F \in C'(I)$.
Also, we have $F'(x) := \frac{d}{dt}\int_a^x f(t)\,dt = f(x) \quad \forall x \in I$.

2) If f is differentiable on I and $f' \in \mathcal{R}(I)$, then
$$\int_a^x f'(t)\,dt = f(x) - f(a).$$

-Notes: a) This tells us that the derivative and integral operations are 'inverses' of one another, and this connection means every rule of one has a corresponding rule for the other.

b) This Thm was the main fact that allowed you to solve (almost) all integral problems in Math 21B.

c) An equivalent version of 2) is:
If $f \in \mathcal{R}(I)$ and there exist a function F which is differentiable on I with $F' = f$, then
$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Thm (5.31) Integration by Parts

Let f, g be differentiable on I where $f', g' \in \mathcal{R}(I)$.

Then, $$\int_a^b f'(x)g(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)dx.$$

- Notes: a) This is connected to the derivative's product rule.

b) The common 'disguise' of this Thm is
$$\int u\,dv = uv - \int v\,du,$$
where $u = g(x)$, $v = f(x)$, $dv = f'(x)dx$, and $du = g'(x)dx$.

Thm (5.34) Change of Variables

Let $\phi \in C'(I)$. If $\phi(x) \neq 0$ $\forall x \in I$ and $f \in \mathcal{R}([c,d])$ where $\phi: I \to [c,d]$, then $(f \circ \phi) \cdot |\phi'| \in \mathcal{R}(I)$, and
$$\int_c^d f(t)dt = \int_a^b f(\phi(x)) \cdot |\phi'(x)|dx$$

- Note: ϕ must be onto, and either a) $\phi(a) = c$, $\phi(b) = d$ or b) $\phi(a) = d$, $\phi(b) = c$.

Thm (5.35) Change of Variables for Continuous Integrands

If $\phi \in C'(I)$ and $f \in C(I^*)$ where I^* is image of ϕ on I, then
$$\int_{\phi(a)}^{\phi(b)} f(t)\,dt = \int_a^b f(\phi(x))\,\phi'(x)dx.$$

- Note: This is just the formula for u-substition learned in 21B, where $u = t = \phi(x)$ and $du = dt = \phi'(x)dx$.