Inverse and Implicit Function Theorems

**Defn** For a differentiable function \( f: \mathbb{R}^n \to \mathbb{R}^n \), the **Jacobian of \( f \) at \( \mathbf{a} \in \mathbb{R}^n \)** is the determinant of the total derivative at \( \mathbf{a} \), which is written as

\[ \Delta_f(\mathbf{a}) := \det(Df(\mathbf{a})). \]

**Lemma (11.39)**

Let \( V \subseteq \mathbb{R}^n \) be open and \( \tilde{f}: V \to \mathbb{R}^n \) be continuous. If \( \tilde{f} \) is 1-1 and has first-order partial derivatives, and if \( \Delta_f(\mathbf{a}) \neq 0 \) on \( V \), then \( \tilde{f}^{-1} \) is continuous on \( \tilde{f}(V) \).

**Lemma (11.40)**

Let \( V \subseteq \mathbb{R}^n \) be open and \( \tilde{f}: V \to \mathbb{R}^n \) with \( \tilde{f} \in C^1(V) \). If \( \Delta_f(\mathbf{a}) \neq 0 \) for some \( \mathbf{a} \in V \), then \( \exists r > 0 \) such that \( B_r(\mathbf{a}) \subseteq V \), \( f \) is 1-1 on \( B_r(\mathbf{a}) \), \( \Delta_f(\mathbf{x}) \neq 0 \quad \forall \mathbf{x} \in B_r(\mathbf{a}) \) and

\[ \det \left[ \frac{\partial f_i}{\partial x_j}(\mathbf{c}_i) \right] \neq 0, \quad \forall \mathbf{c}_1, ..., \mathbf{c}_n \in B_r(\mathbf{a}). \]

**Thm (11.41) The Inverse Function Theorem**

Let \( V \subseteq \mathbb{R}^n \) be open and \( \tilde{f}: V \to \mathbb{R}^n \) with \( \tilde{f} \in C^1(V) \). If \( \Delta_f(\mathbf{a}) \neq 0 \) for some \( \mathbf{a} \in V \), then there exists an open set \( W \) where \( \tilde{a} \in W \) and the following hold:

1) \( \tilde{f} \) is 1-1 on \( W \),
2) \( \tilde{f}^{-1} \in C^1(\tilde{f}(W)) \),
3) \( \forall \mathbf{y} \in \tilde{f}(W) \), \( \tilde{D}(\tilde{f}^{-1}(\mathbf{y}))(\mathbf{	ilde{y}}) = [\tilde{D}\tilde{f}(\tilde{f}^{-1}(\mathbf{y}))]^{-1} \),

where \( \tilde{D}^{-1} \) is matrix inversion.

**Note:** This is a local existence Thm, and for most functions global existence is unattainable.
The Implicit Function Theorem

Let \( V \subseteq \mathbb{R}^{n+p} \) be open and \( \vec{F} = (F_1, \ldots, F_n): V \to \mathbb{R}^n \) with \( \vec{F} \in C^1(V) \). Also, suppose \( \vec{F}(\vec{x}_0, \vec{t}_0) = \vec{0} \) for some pair \((\vec{x}_0, \vec{t}_0) \in V\), where \( \vec{x}_0 \in \mathbb{R}^n \) and \( \vec{t}_0 \in \mathbb{R}^p \).

If
\[
\frac{\partial (F_1, \ldots, F_n)}{\partial (x_1, \ldots, x_n)} (\vec{x}_0, \vec{t}_0) \neq 0, \tag{1}
\]

then \( \exists \ W \subseteq \mathbb{R}^p \) which is open and \( \vec{t}_0 \in W \), and \( \exists ! \ g: W \to \mathbb{R}^n \) with \( g \in C^1(W) \) such that \( g(\vec{t}_0) = \vec{x}_0 \) and \( \vec{F}(\vec{x}(\vec{t}), \vec{t}) = \vec{0} \) \( \forall \vec{t} \in W \).

Note: The LHS of (1) is defined as
\[
\frac{\partial (F_1, \ldots, F_n)}{\partial (x_1, \ldots, x_n)} (\vec{a}) := \det \left[ \frac{\partial F_i}{\partial x_j} (\vec{a}) \right] = \det \left[ \begin{array}{cccc}
\frac{\partial F_1}{\partial x_1} (\vec{a}) & \cdots & \frac{\partial F_1}{\partial x_n} (\vec{a}) \\
\vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial x_1} (\vec{a}) & \cdots & \frac{\partial F_n}{\partial x_n} (\vec{a}) \\
\end{array} \right],
\]

where \((\vec{a}) = (\vec{x}_0, \vec{t}_0)\). This is referred as the partial Jacobian of \( \vec{F} \) and the book has a more general definition of this concept (pg. 430).