Math 125B
Vogler

Inverse and Implicit Function Theorems

Defn For a differentiable function $f: \mathbb{R}^n \to \mathbb{R}^n$, the Jacubian of $\vec{f}$ at $\vec{a} \in \mathbb{R}^n$ is the determinant of the total derivative at $\vec{a}$, which is written as

$$\Delta_{\vec{f}}(\vec{a}) := \det(D\vec{f}(\vec{a})).$$

Lemma (11.39)

Let $V \subseteq \mathbb{R}^n$ be open and $\vec{f}: V \to \mathbb{R}^n$ be continuous. If $\vec{f}$ is 1-1 and has first-order partial derivatives, and if $\Delta_{\vec{f}}(\vec{a}) \neq 0$ on V, then f^{-1} is continuous on $\vec{f}(V)$.

Lemma (11.40)

Let $V \subseteq \mathbb{R}^n$ be open and $\vec{f}: V \to \mathbb{R}^n$ with $\vec{f} \in C'(V)$. If $\Delta_{\vec{f}}(\vec{a}) \neq 0$ for some $\vec{a} \in V$, then $\exists r > 0$ such that $B_r(\vec{a}) \subseteq V$, f is 1-1 on $B_r(\vec{a})$, $\Delta_{\vec{f}}(x) \neq 0 \;\; \forall \vec{x} \in B_r(\vec{a})$ and

$$\det\left[\frac{\partial f_i}{\partial x_j}(\vec{c}_i)\right]_{n \times n} \neq 0, \quad \forall c_1, \dots, c_n \in B_r(\vec{a}).$$

Thm (11.41) The Inverse Function Theorem

Let $V \subseteq \mathbb{R}^n$ be open and $\vec{f}: V \to \mathbb{R}^n$ with $\vec{f} \in C'(V)$. If $\Delta_{\vec{f}}(\vec{a}) \neq 0$ for some $\vec{a} \in V$, then there exists an open set W where $\vec{a} \in W$ and the following hold:

1) $\vec{f}$ is 1-1 on W, 2) $\vec{f}^{-1} \in C'(\vec{f}(W))$,

3) $\forall \vec{y} \in \vec{f}(W)$, $\quad D(\vec{f}^{-1})(\vec{y}) = [D\vec{f}(\vec{f}^{-1}(\vec{y}))]^{-1}$,

where $[\;]^{-1}$ is matrix inversion.

-Note: This is a local existance Thm, and for most functions global existance is unattainable.

Thm (11.47) The Implicit Function Theorem

Let $V \subseteq \mathbb{R}^{n+p}$ be open and $\vec{F} := (F_1, \dots, F_n): V \to \mathbb{R}^n$ with $\vec{F} \in \mathcal{C}^1(V)$. Also, suppose $\vec{F}(\vec{x}_0, \vec{t}_0) = \vec{0}$ for some pair $(\vec{x}_0, \vec{t}_0) \in V$, where $\vec{x}_0 \in \mathbb{R}^n$ and $\vec{t}_0 \in \mathbb{R}^p$.

If

$$\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}(\vec{x}_0, \vec{t}_0) \neq 0, \qquad (1)$$

then $\exists\, W \subseteq \mathbb{R}^p$ which is open and $\vec{t}_0 \in W$, and $\exists!\, \vec{g}: W \to \mathbb{R}^n$ with $\vec{g} \in \mathcal{C}^1(W)$ such that $\vec{g}(t_0) = x_0$ and $\vec{F}(\vec{g}(t), \vec{t}) = \vec{0} \quad \forall \vec{t} \in W$.

- Note: The LHS of (1) is defined as

$$\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}(\vec{a}) := \det\left[\frac{\partial F_i}{\partial x_j}(\vec{a})\right]_{n \times n} = \det\begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial F_1}{\partial x_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial F_n}{\partial x_n}(\vec{a}) \end{bmatrix},$$

where $(\vec{a}) = (\vec{x}_0, \vec{t}_0)$. This is referred as the <u>partial Jacobian of $\vec{f}$</u>, and the book has a more general definition of this concept (pg. 430).