Math 125B

Vogler

Properties of the Integral

Defn Let $I=[a,b]$ and $P=\{x_0, x_1, \dots, x_n\}$ be a partition of I. Also, let $I_j=[x_{j-1}, x_j]$, $j=1,\dots,n$, be a subdivision of I, and $f: I \to \mathbb{R}$. Then,

1) A Riemann sum of f with respect to P generated by sample points $t_j \in I_j$ is the sum

$$S(f,P,t_j) := \sum_{j=1}^{n} f(t_j)\Delta x_j .$$

2) These Riemann sums converge to $I(f)$ as $\|P\| \to 0$ if and only if $\forall \varepsilon > 0$, $\exists$ partition P_ε of I such that $\forall$ partitions P where $P_\varepsilon \subseteq P \Rightarrow |S(f,P,t_j) - I(f)| < \varepsilon$ $\forall$ sample points $t_j \in I_j$, $j=1,\dots,n$. In this case, we denote

$$I(f) := \lim_{\|P\|\to 0} S(f,P,t_j) := \lim_{\|P\|\to 0} \sum_{j=1}^{n} f(t_j)\Delta x_j .$$

-Notes: a) the most common sample points t_j are (as seen in 21B):
i) left endpoints: $t_j = x_{j-1}$ ii) right endpoints: $t_j = x_j$ iii) midpoints: $t_j = \frac{x_j + x_{j-1}}{2}$.
b) This definition can be more useful than the previous one (using upper/lower sum) because it one approximation sum instead of two. (c) Note $\lim_{\|P\|\to 0}$ is equivalent to $\lim_{n\to\infty}$.

Thm (5.18) f is integrable on I if and only if $I(f)$ exists, in which case $I(f) = \int_a^b f(x)dx$.

-Note: This combined with Thm 5.15 tells us all 3 notions of integrability (definition, upper/lower integrals match, and convergence of Riemann sums) are equivalent, and we are free to use which ever is the most convenient.

Thm (5.26) If f is integrable on I, then $F(x) = \int_a^x f(x)dx$ exists and is continuous on I.

<u>Thm (5.19)</u> <u>Linear Property</u> Let $\alpha \in \mathbb{R}$.

If f, g are integrable on I, then $f+g$ and αf are integrable on I. Moreover,

$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \quad \& \quad \int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

<u>Thm (5.20)</u> If f is integrable on I, then f is integrable on each subinterval of I. Also,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall c \in (a,b).$$

<u>Thm (5.21)</u> <u>Comparison Thm of Integrals</u>

If f, g are integrable on I and $f(x) \le g(x)$ $\forall x \in I$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

In particular, if $m \le f(x) \le M$ $\forall x \in I$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

-Note: The left and right side of this last inequality is just the lower and upper Riemann sums with 1 rectangle, respectively.

<u>Thm (5.22)</u> If f is integrable on I, then so is $|f|$ and

$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

<u>Cor (5.23)</u> If f and g are integrable on I, then so is fg.

<u>Thm (5.24)</u> First Mean Value Thm for Integrals.

Let $f \& g$ be integrable on I with $g(x) \ge 0$ $\forall x \in I$. If $m = \inf_{x \in I} f(x)$ and $M = \sup_{x \in I} f(x)$, $\exists c \in [m, M]$ such that

$$\int_a^b f(x)g(x)\,dx = c \int_a^b g(x)\,dx.$$

In particular, if f is continuous on I, then $\exists x_0 \in I$ where

$$\int_a^b f(x)g(x)\,dx = f(x_0) \int_a^b g(x)\,dx.$$

<u>Thm (5.27)</u> Second Mean Value Thm for Integrals

Let $f \& g$ be integrable on I with $g(x) \ge 0$ $\forall x \in I$. If $\exists m, M$ where $m \le \inf_{x \in I} f(x)$ and $M \ge \sup_{x \in I} f(x)$, then $\exists c \in I$ such that

$$\int_a^b f(x)g(x)\,dx = m \int_a^c g(x)\,dx + M \int_c^b g(x)\,dx.$$