Math 125B
Vogler

The Riemann Integral

Defn Let $a < b$.

1) A partition of the interval $I := [a,b]$ is the set of points $P = \{x_0, x_1, \dots, x_n\}$ such that
$$a = x_0 < x_1 < \dots < x_n = b.$$

2) The norm of a partition $P = \{x_0, \dots, x_n\}$ is
$$\|P\| := \max_{1 \le j \le n} (x_j - x_{j-1}) > 0.$$

3) A refinement of a partition P on $[a,b]$ is a partition Q on $[a,b]$ where $P \subseteq Q$. In this case, we say Q is finer than P, or P is coarser than Q.

- Notes: a) Every partition of $I := [a,b]$ decomposes I into n subintervals $I_j := [x_{j-1}, x_j]$, $j = 1, 2, \dots, n$, where $I_j \cap I_k = x_j$ if and only if $k = j+1$ and is empty for $k \ne j$ or $k \ne (j+1)$. Some people call this a subdivision of I. Notice there are always $(n+1)$ points in a partition while it creates n subintervals.

b) Some people call the norm the mesh of the partition.

c) The symbol $\Delta x_j := x_j - x_{j-1}$ represents the length of the jth subinterval, and the symbol Δx will used to represent the length of a uniform subdivision (i.e. $\Delta x_i = \Delta x_j \ \forall i, j = 1, \dots, n$)

d) $\sum_1^n \Delta x_j = b - a$ by a telescoping sum.

<u>Defn</u> Given a function $f: I \to \mathbb{R}$ that is bounded, and the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $I = [a,b]$ let $I_j := [x_{j-1}, x_j]$, $M_j = \sup_{x \in I_j} f(x)$, and $m_j := \inf_{x \in I_j} f(x)$ for $j = 1, 2, \ldots, n$. Then,

1) The upper Riemann sum of f over P is
$$U(f,P) := \sum_{j=1}^{n} M_j \Delta x_j$$

2) The lower Riemann sum of f over P is
$$L(f,P) := \sum_{j=1}^{n} m_j \Delta x_j$$

-Notes: a) Since f is bnded, M_j and m_j exist and are finite $\forall j$.
b) If f is continuous, then it is bounded on I.
c) If $f(x) = \alpha$ (a constant), then $U(f,P) = L(f,P) = \alpha(b-a)$.
d) $L(f,P) \leq U(f,P)$; e) If Q is finer than P then $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$.

<u>Defn</u> A function $f: I \to \mathbb{R}$ is Riemann integrable on $I = [a,b]$ if f is bounded and $\forall \varepsilon > 0$, $\exists$ partition P of I such that $U(f,P) - L(f,P) < \varepsilon$.

<u>Thm (5.10)</u> If f is continuous on I, then it is integrable on I.

<u>Defn</u> Let $f: [a,b] \to \mathbb{R}$ be bounded, and $\mathcal{P}$ be the set of all partitions on $[a,b]$. Then,

1) The upper integral of f on $[a,b]$ is $\overline{\int_a^b} f(x)\,dx := \inf_{P \in \mathcal{P}} U(f,P)$.
2) The lower integral of f on $[a,b]$ is $\underline{\int_a^b} f(x)\,dx := \sup_{P \in \mathcal{P}} L(f,P)$.
3) If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, then we define the integral to be
$$\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

-Note: $\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx$.

<u>Thm (5.15)</u> f is Riemann integrable if and only if the integral exists.

<u>Thm (5.16)</u> If $f(x) = \alpha$ (a constant), then $\int_a^b f(x)\,dx = \alpha(b-a)$.