Inverse Function Theorem

Let $V \subseteq \mathbb{R}^n$ be open and $\vec{f} : V \to \mathbb{R}^n$ with $\vec{f} \in C^{1}(V)$. If $\Delta \vec{f}(\vec{a}) \neq 0$ for some $\vec{a} \in V$, then there exists an open set $W \subseteq V$ where $\vec{a} \in W$ and the following holds:

1) $\vec{f}$ is 1-1 on $W$,
2) $\vec{f}^{-1} \in C^{1}(\vec{f}(W))$,
3) $\forall \vec{y} \in \vec{f}(W)$, $D(\vec{f}^{-1})(\vec{y}) = [D \vec{f}(\vec{f}^{-1}(\vec{y}))]^{-1}$.

Proof. By Lemma 11.40, $\exists \ r > 0$ such that for the open ball $B := B_{\mathcal{r}}(\vec{a})$, we have $\vec{f}$ is 1-1 and $\Delta \vec{f} \neq 0$ on $B$. Also, we have $\Delta := \det \left[ \frac{\partial f_i}{\partial x_j}(\vec{c}) \right] \neq 0 \ \forall \vec{c} \in B$.

By Lemma 11.39, $\vec{f}^{-1} \in C^{1}(\vec{f}(B))$. Note $B \subseteq V$ by Lemma 11.40.

Choose $W \subseteq B \subseteq V$ to be an open ball centered at $\vec{a}$, but smaller than $B$ (i.e. $W = B_{\mathcal{r}_0}(\vec{a})$ with $0 < \mathcal{r}_0 < \mathcal{r}$). Then, $\vec{f}$ is 1-1 on $W$, proving claim 1). Also, $\vec{f}(W)$ must be open since $\vec{f}^{-1}$ is continuous on $\vec{f}(W)$ by Lemma 11.39.

We want to show the first partials of $\vec{f}^{-1}$ exist and are continuous on $\vec{f}(W)$. Fix $\vec{y} \in \vec{f}(W)$ along with $i, k = 1, \ldots, n$. Choose $\epsilon \in \mathbb{R} \setminus \{0\}$ small enough where $\vec{f}^{-1}(\vec{y}) + \epsilon \vec{e}_k \in \vec{f}(W)$. Let $\vec{x} = f^{-1}(\vec{y})$ and $\vec{w} = f^{-1}(\vec{y} + \epsilon \vec{e}_k)$.

Notice, for each $i = 1, \ldots, n$, we have

$$f_i(\vec{w}) - f_i(\vec{x}) = \begin{cases} t & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

or

$$\frac{f_i(\vec{w}) - f_i(\vec{x})}{t} = \begin{cases} 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$
Thus, by Mean Value Thm (Thm 11.30), there exists n points (one for each i) \( \tilde{z}_i \in L(\tilde{x}; \tilde{w}) \) where

\[
\nabla f_i(\tilde{z}_i) \cdot \frac{\hat{x} - \tilde{w}}{t} = \frac{f_i(\hat{x}) - f_i(\tilde{w})}{t} = \begin{cases} 
1 & \text{if } k = i \\
0 & \text{if } k \neq i 
\end{cases} \quad \text{for } i = 1, \ldots, n \tag{1}
\]

Define \( \vec{z} = \frac{\hat{x} - \tilde{w}}{t} \) and matrix \( A = \left[ \frac{\partial f_i}{\partial x_j}(c_i) \right]_{n \times n} \), so (1) becomes

\[
A \vec{z} = \begin{bmatrix}
0 \\
\vdots \\
1 \\
0
\end{bmatrix}^k = : \vec{r} \tag{2}
\]

Equation (2) is a system of \( n \) linear equations and \( n \) unknowns (i.e., \( \frac{x_j - w_j}{t} \)) where the coefficient matrix \( A \) has determinant \( \Delta \neq 0 \) on \( W \subseteq B \) (by our choice of \( B \)), Using Cramer's Rule (see Appendix C), solutions to (2) satisfy

\[
(t^{-1})_j (\vec{y} + t \hat{e}_k) - (t^{-1})_j (\vec{y}) = \frac{x_j - w_j}{t} = Q_j(t) = \frac{\det(A(i))}{\Delta}, \tag{3}
\]

where \( A(i) \) is the matrix formed by replacing the \( j \)th column of \( A \) with the vector \( \vec{r} \). Notice \( Q_j(t) \) is the quotient of determinants for matrices which have entries of \( 0, 1 \), or first-order partials evaluated at the \( \tilde{z}_i \)'s. When \( t \to 0 \), we have \( \tilde{w} \to \hat{x}, \tilde{z}_i \to \hat{x}, \forall i = 1, \ldots, n \), and \( \vec{y} + t \hat{e}_k \to \vec{y} \). Also, \( Q_j(t) \to Q_j \), which is a quotient of determinants whose entries consist of \( 0, 1 \), or first-order partials evaluated at \( \vec{x} = f^{-1}(\vec{y}) \).
Since $\tilde{f}^{-1}$ is continuous on $\tilde{f}(\mathbf{w})$, $Q_j$ must be continuous $\forall \tilde{y} \in \tilde{f}(\mathbf{w})$ through algebraic and functional composition of continuous functions. Taking the limit of (3) as $t \to 0$, we have

$$\frac{\partial (f^{-1})}{\partial x_k} (\tilde{y}) = Q_j \quad \forall j=1,\ldots,n, \forall \tilde{y} \in \tilde{f}(\mathbf{w}). \quad (4)$$

Since $k=1,\ldots,n$ was arbitrary, (4) holds for all $k=1,\ldots,n$. Therefore, all 1st-order partials of $f^{-1}$ exist on $\tilde{f}(\mathbf{w})$, and we conclude $\tilde{f}^{-1} \in C^1(\tilde{f}(\mathbf{w}))$, proving claim 2).

For claim 3), fix $\tilde{y} \in \tilde{f}(\mathbf{w})$. By HW 11.2.8 and Chain Rule, we can directly compute

$$I = D[I\tilde{y}] = D[(f \circ f^{-1})(\tilde{y})] = D\tilde{f}(\tilde{f}^{-1}(\tilde{y})):D\tilde{f}^{-1}(\tilde{y}),$$

where $I$ is the n$x$n identity matrix. By the uniqueness of matrix inverses, we conclude

$$D\tilde{f}^{-1}(\tilde{y}) = [D\tilde{f}(\tilde{f}^{-1}(\tilde{y}))]^{-1},$$

proving claim 3). $\blacksquare$